Start with \$1.

Invest $\$x$ in a risky asset today; the value of the asset multiplies by $\gamma$ overnight. Tomorrow invest a total of $\$y$ in the risky asset. $y$ is allowed to depend on $\gamma$. The asset multiplies by $\delta$ the second night, with $\gamma$ and $\delta$ i.i.d.

At the interest rate $r$, your assets at the end are

$$((1-x)\cdot(1+r)+x\cdot\gamma-y)\cdot(1+r)+y\cdot\delta.$$

Consider the case where $r$ =0 and set $\alpha=\gamma-1$, $\beta=\delta-1$. Then your increase in assets from period zero to period two is given by the expression

$$x\cdot\alpha+y(\alpha)\cdot\beta. \tag{1}$$

Write $\mu$ for the mean of $\alpha$ (and $\beta$) and $\sigma$ for the variance of $\alpha$ (and $\beta$).

A *strategy* consists of a number $x$ (the initial investment) and a function $y(\alpha)$.

The goal, for a fixed expected value of (1), is to minimize the variance of (1). Fix the expected value at $c$. Then we always have

$$x=\frac{c}{\mu}-\bar{y}. \tag{2}$$

Now the problem is to choose a function $y(\alpha)$, such that when $x$ is determined by (2), the variance of (1) is minimized.

Note that

$$Var(x\cdot\alpha+y\cdot\beta)=Var(x\cdot\alpha)+Var(y\cdot\beta)+2\cdot Cov(x\cdot\alpha,y\cdot\beta)$$

and that

$$Cov(x\cdot\alpha,y\cdot\beta)=x\cdot\mu\cdot Cov(\alpha,y)$$

by the independence assumptions, so that

$$Var(x\cdot\alpha+y\cdot\beta)=x^2\cdot\sigma+\bar{y}^2\cdot\sigma+\mu^2\cdot Var(y)+\sigma\cdot Var(y)+2\cdot x\cdot\mu\cdot Cov(\alpha,y) \tag{3}$$

where $x$ continues to be given by expression (2).

Write $E=\bar{y}$, $V=Var(y)$, and $C=Cov(\alpha,y)$. Then using (2), the right-hand side of (3) becomes

$$(\frac{c}{\mu})^2\sigma-\frac{2c\sigma}{\mu}E+2\sigma E^2+(\mu^2+\sigma)V+2cC-2\mu EC. \tag{4}$$

**Experiment 1:** Try adding a constant $t$ to $y$. This replaces $E$ by $E+t$ while leaving $V$ and $C$ fixed. Insert this information into (4), differentiate with respect to $t$, and set $t=0$. The resulting expression must equal zero if $y$ is optimal. This gives

$$E=\frac{c\sigma+\mu^2C}{2\sigma\mu}. \tag{5}$$

**Experiment 2:** Try multiplying $y$ by a constant $t$. This replaces $E$ by $tE$, $V$ by $t^2V$ and $C$ by $tC$. Insert into (4), differentiate with respect to $t$, set $t=1$ and note that the expression must equal zero.

But the expression just described is

$$-\frac{2c\sigma}{\mu}E+4\sigma E^2+2(\mu^2+\sigma)V+2cC-4\mu EC$$

which, after taking account of (5), reduces to

$$2(\mu^2+\sigma)V.$$

Assuming $\sigma\neq 0$, we conclude that $V=0$, so $y$ is a constant, so $C=0$, so (5) gives

$$y = \frac{c}{2\mu}$$

and (1) gives

$$y = x.$$

So to minimize risk, the total amount invested in period 2 should equal the total amount invested in period 1, independent of the realized return in period 1.

Remaining questions: What if $r$ not zero? What if you are simultaneously trying to smooth consumption? What if attitudes toward risk depend on wealth and hence on $\alpha$? What happens in equilibrium? If everyone tries to follow this strategy, how does the interest rate behave in RE equilibrium?