Let X be a regular scheme.

Let X^n be the n-simplex over X, that is

$$X^n = X \times_{\mathbf{Z}} \mathbf{Z}[t_0, \ldots, t_n]/(\sum_{i=0}^{n} t_i) - 1)$$

Let $\mathcal{S}_i$ be the set of all strata of codimension i in X^n, where a "stratum" is an intersection of irreducible components. For $S \in \mathcal{S}_i$, let $j : S \to X^n$ be the inclusion, and let $\mathcal{K}_m(S)$ denote the sheaf of K_m groups on S. Then in the category of sheaves on X^n we have an obvious complex

$$0 \to \mathcal{K}_m \to \bigoplus_{S \in \mathcal{S}_0} j_* \mathcal{K}_m(S) \to \bigoplus_{S \in \mathcal{S}_1} j_* \mathcal{K}_m(S) \to \cdots \to \bigoplus_{S \in \mathcal{S}_n} j_* \mathcal{K}_m(S) \to 0$$

Call this complex $\mathcal{K}_m$. (The notation suppresses the dependence on n.)

Now let $\mathcal{X}_n$ be the nerve of the covering of X^n by its strata. Let Φ be any family of supports on X^n. Then there is a hypercohomology spectral sequence

$$E_2^{p,q} = H^{p+q}(\mathcal{X}^n, H^q_\Phi \mathcal{K}_m) \Rightarrow \mathbf{H}^p_\Phi(X^n, \mathcal{K}_m) \tag{1}$$

Here $H^q_\Phi \mathcal{K}_m$ means the system of coefficients $S \mapsto H^q_\Phi(S, \mathcal{K}_m(S))$. The notation $H^*_\Phi(Y, -)$ will always mean cohomology with supports in $\Phi \cap Y$.

Now let S be a stratum of X^n and let $S(p)$ be the set of points in S of codimension p. We have the Gersten resolution

$$0 \to \mathcal{K}_m(S) \to \bigoplus_{y \in S_0} \mathcal{K}_m k(y) \to \bigoplus_{y \in S_1} \mathcal{K}_m k(y) \to \cdots \bigoplus_{y \in S_m} \mathcal{K}_m k(y)$$

From this resolution, we see that if Φ is any family of supports with pure codimension m, then $H^m_\Phi(S, \mathcal{K}_m)$ is just the free abelian group $G(\Phi, S)$ on the components of $S \cap \Phi$. It is also immediate that $H^*_\Phi(S, \mathcal{K}_m) = 0$ for $* \neq m$.

Now specialize to the case where Φ consists of all closed irreducible sets of codimension m in X^n. It follows from the computations in the preceding paragraph that the only non-zero E_2 terms in the spectral sequence (1) occur on the row $q = m$, and in particular we have

$$\mathbf{H}^m_\phi(\mathcal{X}^n, \mathcal{K}_m) = H^0(\mathcal{X}^n, H^m_\Phi \mathcal{K}_m) = \tilde{Z}^m(X, n) \tag{2}$$

where $\tilde{Z}^m(X, n)$ is the kernel in the sequence

$$0 \to \tilde{Z}^m(X, n) \to \bigoplus_{S \in \mathcal{S}_0} Z^m(S) \to \bigoplus_{S \in \mathcal{S}_1} Z^m(S)$$

and $Z^m(S)$ is the group of codimension m cycles on S.

Next we use the spectral sequence (1) without supports to compute $\mathbf{H}^m(X, \mathcal{K}_m)$. A theorem of Clay Sherman says that

$$H^q(S, \mathcal{K}_m) = H^q(X, \mathcal{K}_m)$$

for all q and m (here we are using the fact that X is regular and S is isomorphic to X crossed with affine space), so that the system of coefficients is constant on the nerve $\mathcal{X}^n$. It follows from this and the fact that the geometric realization of $\mathcal{X}^n$ is a sphere that the E_2 terms are non-zero only for $p = 0, n$ and we have

$$E_2^{0,q} = E_2^{n,q} = H^q(X, \mathcal{K}_m)$$

Thus the spectral sequence gives

$$0 \to H^{m-1}(X, \mathcal{K}_m) \to H^{m-n}(X, \mathcal{K}_m) \to \mathbf{H}^m(X, \mathcal{K}_m) \to H^m(X, \mathcal{K}_m) \tag{3}$$

Note that the rightmost term in the above exact sequence is known ("Bloch's Formula") to be $Ch^m(X)$.

Finally, consider the map from cohomology with supports to cohomology without supports:

$$\begin{array}{ccccccccc}
 & & & & \tilde{Z}^m(X,n) & & & & \\
 & & & & \| & & & & \\
 & & & & \mathbf{H}^m_\Phi(X^n, m) & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & H^{m-1}(X, \mathcal{K}_m) & \xrightarrow{d} & H^{m-n}(X, \mathcal{K}_m) & \to & \mathbf{H}^m(X^n, \mathcal{K}_m) & \to & H^m(X, \mathcal{K}_m) & \to & 0 \\
 & & & & & & & & \| & & \\
 & & & & & & & & Ch^m(X) & &
\end{array}$$

The kernel of the vertical map is an image of $\mathbf{H}^{m-1}(X^n - \bigcup \Phi, \mathcal{K}_m)$, which can be identified with the image of $Z^{m+1}(X \times \mathbf{A}^{n+1})$ in $\tilde{Z}^m(X,n)$. The quotient of $\tilde{Z}^m(X,n)$ by this image is by definition the higher Chow group $Ch^m(X,n)$.

The map from the higher Chow group $Ch^m(X,n)$ to the lower Chow group $Ch^m(X)$ is clearly zero.

Therefore we get a map

$$Ch^m(X,n) \to H^{m-n}(X, \mathcal{K}_m)/\text{image}(d)$$

.

I assume there is some obvious way to lift this map to $H^{m-n}(X, \mathcal{K}_m)$ but I don't quite see it right now. I also assume this is the same map I construct in my paper on Relative Chow Groups, but I don't remember whether I ever fully checked that.