III. Geometry

In topology we studied manifolds; in geometry we study manifolds equipped with metrics (to be defined in III.1). In topology we studied operators that depend on first derivatives (i.e. tangent vectors); in geometry we study operators that depend on second derivatives (e.g. the curvature tensor, to be introduced in III.3). In principle, there is room for argument about whether the transition from topology to geometry is marked by the introduction of the metric or the shift in emphasis from first to second derivatives. In pratice, at least for our purposes, the two positions are more or less equivalent.

1. Metrics.

A. Basic Definitions

Reminders 1.1. Let M be a manifold. Let $T^{0,2}M$ be the bundle of $(0,2)$-tensors on M (II.5.5). Let $\Gamma(M, T^{0,2}M)$ be the set of sections (II.2.26) of $T^{0,2}M$. Then we have

$$\Gamma(M, T^{0,2}M) \approx \Gamma(M, T^*M \otimes T^*M) \tag{1.1.1}$$

$$\approx \Gamma(M, Hom(T_*M, T^*M)) \tag{1.1.2}$$

$$\approx \mathrm{Hom}_{VB}(T_*M, T^*M) \tag{1.1.3}$$

where the identification (1.1.1) is from (II.5.4), the identification (1.1.2) is from (I.4.13.1) combined with (II.3.23), and the identification (1.1.3) is from (II.3.14).

Definition 1.2. Let $\mathbf{g} \in \Gamma(M, T^{0,2}M)$. Then $\mathbf{g}$ is *nondegenerate* if its image in $Hom_{VB}(T_*M, T^*M)$ is an isomorphism of vector bundles.

Remark 1.3. According to (II.3.24), a vector bundle need not in general be isomorphic to its dual; in particular, the tangent bundle T_*M need not be isomorphic to the cotangent bundle T^*M. The existence of an isomorphism $T_*M \to T^*M$ is (clearly) equivalent to the existence of a nondegenerate section of $T^{0,2}M$.

Notation 1.4.. Given $\mathbf{g} \in \Gamma(M, T^{0,2}M)$ and given $m \in M$, we have (from the definition of a section) $\mathbf{g}(m) \in T^{0,2}_m M$. We will usually write $\mathbf{g}_m$ instead of $\mathbf{g}(m)$.

Exercise 1.5. Let M be a manifold and let $\mathbf{g}$ be a section of $T^{0,2}M$. Show that the following two conditions are equivalent:

i) $\mathbf{g}$ is nondegenerate (1.2)

ii) For each $m \in M$, $\mathbf{g}_m \in T^{0,2}_m M$ is nondegenerate (I.6.3).

Definition 1.6. Let M be a manifold and let $\mathbf{g}$ be a section of $T^{0,2}M$. We say that $\mathbf{g}$ is *symmetric* if $\mathbf{g}(m) \in T^{0,2}_m M$ is symmetric (I.6.2) for every $m \in M$.

Definition 1.7. A section $\mathbf{g} : M \to T^{0,2}M$ is called a *metric on* M if it is nondegenerate and symmetric.

Exercise 1.7.1. Show that $\mathbf{g}$ is a metric (1.6) if and only if each $\mathbf{g}_m$ is an inner product (I.6.4).

Blanket Assumption 1.8. Henceforth, M represents a manifold with a metric $\mathbf{g}$. Sometimes we will write $(M, \mathbf{g})$ when it seems advisable to stress which metric we're talking about. We will use the same symbol $\mathbf{g}$ to denote the associated isomorphism

$$\mathbf{g} : T_*M \xrightarrow{\approx} T^*M \tag{1.8.1}$$

Definitions and Notation 1.9. Recall from (II.6.4) and (II.6.1) that a *vector field* is a section $X : M \to T_*M$ and a *one-form* is a section $\xi : M \to T^*M$.

Given a vector field X, we define the *associated one-form* to be the composition $\mathbf{g} \circ X$, and given a one-form ξ we define the *associated vector field* to be the composition $\mathbf{g}^{-1} \circ \xi$ where $\mathbf{g}$ represents the isomorphism (1.8.1).

Now let $U \subset M$ be open and recall from (II.2.27) that $\mathcal{C}(U)$ denotes the set of smooth real-valued functions on U.

Given a vector field $X : M \to T_*M$, and a vector field $Y : U \to T_*U \subset T_*M$, we define

$$< X, Y > = < \mathbf{g} \circ X, Y > : U \to \mathbf{R}$$

where $< \mathbf{g} \circ X, Y >$ is as defined in (II.3.17).

We also define

$$< X, - >: \quad \begin{array}{ccc} \Gamma(U, T_*M) & \to & \mathcal{C}(U) \\ Y & \mapsto & < X, Y > \end{array} \tag{1.9.1}$$

Exercises 1.10. i) Show that every one-form is associated to a unique vector field, and every vector field is associated to a unique one-form.

ii) Given $m \in M$, identify $\mathbf{g}_m \in T_m^{0,2}M$ with a bilinear map

$$\mathbf{g}_m : T_mM \times T_mM \to \mathbf{R}$$

as in (I.6.1). Show that for any two vector fields X and Y, we have

$$< X, Y > (m) = \mathbf{g}_m(X(m), Y(m)) \in \mathbf{R} \tag{1.10.1}$$

iii) Use the symmetry of $\mathbf{g}$ to show that $< X, Y >=< Y, X >$.

iv) For vector fields X, Y, and Z, and for a smooth function $f : M \to \mathbf{R}$, show that

$$< fX + Y, Z >= f < X, Z > + < Y, Z >$$

Remark 1.10.2. We could have used (1.10.1) to *define* $< X, Y >$, but then we would have had to check that the functions $< X, Y >: M \to \mathbf{R}$ and $< X, - >: M \to T^*M$ are smooth. This would have required us to reconstruct the definitions in (1.9).

Notation 1.11. Suppose M is parallelizable (II.6.6) and let $X_1, \ldots, X_n$ be a global basis for M (II.6.7). Define

$$\begin{array}{llll} X_i \otimes X_j : & M & \longrightarrow & T^{0,2}M \\ & & & \cup \\ & m & \mapsto X_i(m) \otimes X_j(m) \in & T_m^{0,2}M \end{array}$$

Proposition 1.12. If M is parallelizable and $X_1, \ldots, X_n$ is a global basis for M, then

$$\mathbf{g} = \sum_{i,j=1}^{n} < X_i, X_j > (X_i \otimes X_j) : M \to T^{0,2}M$$

Proof. This is an exercise in chasing through the definitions.

Remark and Notation 1.13. In view of (1.12), $\mathbf{g}$ is entirely determined by the n^2 smooth functions $< X_i, X_j >$. But these functions cannot be arbitrary, because the symmetry condition implies that $< X_i, X_j >=< X_j, X_i >$ and the nondegeneracy condition imposes additional constraints.

We will say that the matrix of smooth functions

$$\begin{pmatrix} < X_1, X_1 > & < X_1, X_2 > & \ldots & < X_1, X_n > \\ < X_2, X_1 > & < X_2, X_2 > & \ldots & < X_2, X_n > \\ \vdots & \vdots & & \vdots \\ < X_n, X_1 > & < X_n, X_2 > & \ldots & < X_n, X_n > \end{pmatrix}$$

represents $\mathbf{g}$ *with respect to the global basis* $\{X_1, \ldots, X_n\}$.

Definition 1.14. The global basis $\{X_1, \ldots, X_n\}$ is called *orthonormal* (with respect to $\mathbf{g}$) if there exists a k, $0 \leq k \leq n$, such that

$$< X_i, X_j >= \begin{cases} -1 & \text{if } i = j \leq k \\ 1 & \text{if } i = j > k \\ 0 & \text{if } i \neq j \end{cases}$$

Definition 1.15. Let X_i be the vector field $\partial/\partial x_i$ on $\mathbf{R}^n$. Let k be an integer, $0 \leq k \leq n$. The *standard metric of signature* k on $\mathbf{R}^n$ is the metric

$$-\sum_{i=1}^{k} X_i \otimes X_i + \sum_{i=k+1}^{n} X_i \otimes X_i$$

where $0 \leq k \leq n$.

The *usual metric* on $\mathbf{R}^n$ is the standard metric of signature 0; i.e.

$$\sum_{i=1}^{n} X_i \otimes X_i$$

Exercise 1.15.1. Show that a metric $\mathbf{g}$ on $\mathbf{R}^n$ is a standard metric if and only if the X_i form an orthonormal basis for $\mathbf{g}$.

Example 1.16. Let x and y be the coordinates on $\mathbf{R}^2$. Consider the vector fields

$$X = cos(x^2+y^2)\frac{\partial}{\partial x} + sin(x^2+y^2)\frac{\partial}{\partial y}$$

$$Y = -sin(x^2+y^2)\frac{\partial}{\partial x} + cos(x^2+y^2)\frac{\partial}{\partial y}$$

(We first met these vector fields in (I.6.22).

It is a trivial consequence of the definition that $\partial/\partial x$ and $\partial/\partial y$ constitute an orthonormal basis for the usual metric on $\mathbf{R}^2$. Show that X and Y constitute another orthonormal basis for the same metric.

B. Induced Metrics.

Definitions 1.17. Let $i : M \to N$ be an injective map of manifolds. Given $m \in M$, write $n = i(m)$. Then (II.4.15) gives a map

$$i_{*m} : T_m M \to T_n N \tag{1.17.1}$$

The map i is called an *imbedding* if for every $m \in M$, the map (1.17.1) is injective.

Remarks and Definition 1.18. If $i : M \to N$ is an imbedding, we can use (1.17.1) to identify $T_m M$ with a subspace of the vector space $T_n N$ where $n = i(m)$. Thus any inner product $\mathbf{g}_n$ on $T_n N$ can be restricted to an inner product $\mathbf{g}_m$ on $T_m M$; explicitly, we can think of $\mathbf{g}_n$ as a bilinear map

$$\mathbf{g}_n : T_n N \times T_n N \to \mathbf{R}$$

and then just restrict this map to $T_m M \times T_m M$.

Alternatively, we can apply the contravariant functor $T^{0,2}$ to (1.17.1) and get a map

$$T_n^{0,2} : T_n^{0,2} N \to T_m^{0,2} M$$

and then define the restriction $\mathbf{g}_m$ to be $T_n^{0,2}(\mathbf{g}_n)$ a la (I.6.5). According to (I.6.7), this definition of the restriction is equivalent to that of the preceding paragraph.

According to the notation of (I.6.5), the restricted inner product should be called $(i_{*m})^*(\mathbf{g}_n)$. We will abbreviate this to $i^*\mathbf{g}_n$.

Now let $\mathbf{g}$ be a metric on N and define a section $i^*\mathbf{g} : M \to T^{0,2}M$ by

$$(i^*\mathbf{g})(m) = T^{0,2}(i_{*m})(\mathbf{g}_{i(m)}) = i^*(\mathbf{g}_{i(m)}) \tag{1.18.1}$$

Then $i^*\mathbf{g}$ is a metric on M, called the *pullback of* $\mathbf{g}$ *to* M. If the injection $i : M \to N$ is an inclusion, we will also call $i^*\mathbf{g}$ the *restriction* of $\mathbf{g}$ to M or the metric *induced* on M by $\mathbf{g}$. In this case we will write $\mathbf{g}|_M$ instead of $i^*\mathbf{g}$.

Example 1.19. Consider the inclusion $i : \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$. The usual metric $\mathbf{g}$ on $\mathbf{R}^3$ induces a metric on $\mathbf{S}^2$. In this example, we will explicitly compute that induced metric.

First, we need to confine our attention to a coordinate patch on $\mathbf{S}^2$. We use the coordinate patch of (II.1.3.7), (II.4.14.3) and (II.4.15.3); that is, $\Omega \subset \mathbf{S}^2$ is the complement of the set $Z = \{(x, y, z) \in \mathbf{S}^2 | x \leq 0, y = 0\}$. As in those earlier examples, we consider the chart ϕ whose inverse is given by

$$\begin{array}{llcc} \phi^{-1}: & (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) & \to & \Omega \\ & (u, v) & \mapsto & (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)) \end{array} \tag{1.19.1}$$

so that v represents "latitude" and u represents "longitude".

We will compute the matrix of $\mathbf{g}$ with respect to the global basis $\{\partial/\partial u^\phi, \partial/\partial v^\phi\}$. (Of course this is not a global basis on $\mathbf{S}^2$, but it is a global basis on the subset Ω to which we have restricted our attention.)

For this we use (II.4.15.3.1) and (II.4.15.3.2), reproduced here for convenience:

$$i_{*m}\left(\frac{\partial}{\partial u^\phi}\right) = -\sin(u^\phi)\cos(v^\phi)\frac{\partial}{\partial x} + \cos(u^\phi)\cos(v^\phi)\frac{\partial}{\partial y}$$

$$i_{*m}\left(\frac{\partial}{\partial v^\phi}\right) = -\cos(u^\phi)\sin(v^\phi)\frac{\partial}{\partial x} - \sin(u^\phi)\sin(v^\phi)\frac{\partial}{\partial y} + \cos(v^\phi)\frac{\partial}{\partial z}$$

Here, as in earlier examples, v^ϕ is shorthand for the function $v \circ \phi$.

Now we compute

$$\begin{aligned}\left\langle \frac{\partial}{\partial u^\phi}, \frac{\partial}{\partial u^\phi} \right\rangle &= \left\langle i_{*m}\left(\frac{\partial}{\partial u^\phi}\right), i_{*m}\left(\frac{\partial}{\partial u^\phi}\right)\right\rangle \\ &= \sin^2(u^\phi)\cos^2(v^\phi) + \cos^2(u^\phi)\cos^2(v^\phi) \\ &= \cos^2(v^\phi)\end{aligned}$$

Here the $<,>$ on the left is defined with respect to the induced metric on Ω and the $<,>$ on the right is defined with respect to the usual metric on $\mathbf{R}^3$; the first equality is the definition of the induced metric and the second equality is computed using $< \partial/\partial x, \partial/\partial x >=< \partial/\partial y, \partial/\partial y >= 1$ and $< \partial/\partial x, \partial/\partial y >=< \partial/\partial y, \partial/\partial x >= 0$.

Similar calculations show that

$$\left\langle \frac{\partial}{\partial u^\phi}, \frac{\partial}{\partial v^\phi} \right\rangle = \left\langle \frac{\partial}{\partial v^\phi}, \frac{\partial}{\partial u^\phi} \right\rangle = 0$$

$$\left\langle \frac{\partial}{\partial v^\phi}, \frac{\partial}{\partial v^\phi} \right\rangle = 1$$

so that the induced metric is represented by the matrix

$$\begin{pmatrix} \cos^2(v^\phi) & 0 \\ 0 & 1 \end{pmatrix} \tag{1.19.2}$$

Exercise 1.19.3. Carry out the "similar calculations" referenced in the preceding paragraph.

Definition 1.20. Suppose M and N are manifolds with metrics $\mathbf{g}_M$ and $\mathbf{g}_N$. An *isometry*

$$i : (M, \mathbf{g}_M) \to (N, \mathbf{g}_N)$$

is a diffeomorphism

$$i : M \to N$$

such that

$$i^*\mathbf{g}_N = \mathbf{g}_M$$

Intuitively, the existence of an isometry means that there is no essential difference between $(M, \mathbf{g}_M)$ and $(N, \mathbf{g}_N)$.

Two manifolds with metrics are *isometric* if there is an isometry from one to the other. You should think of isometric manifolds as essentially indistinguishable.

Definition 1.21. Let $f : M \to N$ be a smooth map with image $\Omega \subset N$. Then

$$f : (M, \mathbf{g}_M) \to (N, \mathbf{g}_N)$$

is called an *isometric imbedding* if Ω is open in N and the map

$$f : (M, \mathbf{g}_M) \to (\Omega, \mathbf{g}_N|_\Omega)$$

is an isometry.

1C. Metrics on Vector Spaces

Discussion 1.22. Let V be a vector space. Then V automatically has the structure of a manifold via (II.1.3.5).

As a vector space, V can be equipped with an inner product. As a manifold, V can be equipped with a metric. In this subsection we will associate a metric to each inner product and an inner product to each metric; that is, we will construct maps (of sets)

$$\left\{ \begin{array}{c} \text{Inner Products on} \\ \text{the Vector Space } V \end{array} \right\} \underset{G}{\overset{F}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Metrics on} \\ \text{the Manifold } V \end{array} \right\}$$

We will see that the composition $G \circ F$ is the identity. The composition $F \circ G$ is *not* the identity. Those metrics $\mathbf{g}$ for which $F(G(\mathbf{g})) = \mathbf{g}$ are in a sense the "simplest" metrics. We will call these metrics *constant metrics* and will develop a vocabulary for measuring the extent to which a metric fails to be constant.

Construction 1.23. Given an inner product g on the vector space V, we want to define a metric $\mathbf{g} = F(g)$ on the manifold V.

For each $m \in M$, we have an isomorphism (II.4.16.3)

$$\begin{array}{rccc} \theta_m : & V & \to & T_m V \\ & v & \mapsto & D_v \end{array}$$

where D_v denotes "differentiation in the direction v". The idea is to transfer the inner product g from V to T_mV along this isomorphism; more precisely, define an inner product $\mathbf{g}_m$ on T_mV by

$$\mathbf{g}_m(D_v, D_w) = g(v, w)$$

Now, thinking of $\mathbf{g}_m$ as an element of $T^{0,2}_m V$, define a section

$$\mathbf{g} = F(g) \in \Gamma(V, T^{0,2}V)$$

by

$$\mathbf{g}(m) = \mathbf{g}_m$$

We will call $F(g)$ the *metric associated with* g.

When more than one vector space is under discussion, we will write $F_V(g)$ instead of $F(g)$ as necessary to avoid confusion.

Definition 1.24. A metric of the form $F(g)$ is called a *constant metric*.

More generally, let U be any open subset of V. Then on U, a metric of the form $F(g)|_U$ is called a *constant metric*.

Exercise 1.25. Let $i : V \to W$ be an isomorphism of vector spaces and let g be an inner product on W. Then

$$F_V(i^*(g)) = i^*(F_W(g))$$

Here $i^*(g)$ is the pullback of the inner product g to the vector space V (I.6.6) and $i^*(F(g))$ is the pullback of the metric $F(g)$ to the manifold V (1.18).

Remarks and Definition 1.26. In (1.28) we will show that constant metrics are particularly simple; here we introduce the vocabulary necessary to state that result:

Let $(V, \mathbf{g})$ and $(W, \mathbf{h})$ be vector spaces with metrics. A map

$$\phi : (V, \mathbf{g}) \to (W, \mathbf{g}) \tag{1.26.1}$$

is called a *linear isometry* if it is both an isomorphism of vector spaces and and isometry.

$(V, \mathbf{g})$ and $(W, \mathbf{h})$ are *linearly isometric* if there exists a linear isometry (1.26.1).

Remark 1.27. Consider the following two conditions:

i) V is isomorphic to W and $(V, \mathbf{g})$ is isometric to $(W, \mathbf{h})$.

ii) $(V, \mathbf{g})$ is linearly isometric to $(W, \mathbf{h})$.

Then clearly (ii) implies (i), but the converse is not true. Thus linear isometry is a very strong form of "looking exactly alike".

Proposition 1.28. Let $\mathbf{g}$ be a constant metric on a vector space V. Then $(V, \mathbf{g}))$ is linearly isometric to $(\mathbf{R}^n, \mathbf{s})$ where $\mathbf{s}$ is a standard metric on $\mathbf{R}^n$ (1.15).

Proof. Write $\mathbf{g} = F(g)$ where g. Write k for the signature of g (I.6.10). Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V, let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbf{R}^n$, and let $j : V \to \mathbf{R}^n$ be the unique linear transformation that takes v_i to e_i for all i. Then $j : (V, \mathbf{g}) \to (\mathbf{R}^n, \mathbf{s})$ is the desired linear isometry.

Construction 1.29. Let $\mathbf{g}$ be a metric on V. Let 0 be the zero element of V and consider the inner product $\mathbf{g}_0$ on T_0V. By (II.4.16.3), there is a natural isomorphism

$$\theta_{V,0} : V \to T_m V$$

which takes an element v to the "directional derivative in the v direction".

Now define $G(\mathbf{g}) : V \otimes V \to \mathbf{R}$ by

$$G(\mathbf{g})(v, w) = \mathbf{g}_0(\theta_{V,0}(v), \theta_{V,0}(w))$$

and call $G(\mathbf{g})$ the *inner product associated with the metric* $\mathbf{g}$.

Remark 1.29.1. It is immediate from the construction that if $\mathbf{g}$ and $\mathbf{h}$ are two metrics with $\mathbf{g}_0 = \mathbf{h}_0$, then $G(\mathbf{g}_0) = G(\mathbf{h}_0)$.

Remark 1.29.2. If $U \subset V$ is an open subset containing 0 and if $\mathbf{g}$ is a metric on U, then we can still use (1.29) to define an inner product $G(\mathbf{g})$ on V.

Exercise 1.29.3. Show that for any inner product g, we have $G(F(g)) = g$.

Exercise 1.29.4. Show that a metric $\mathbf{g}$ is constant if and only if $F(G(\mathbf{g})) = \mathbf{g}$.

Discussion 1.30. Given a metric $\mathbf{g}$, we want to develop a measure of how far $\mathbf{g}$ is from being constant. First we need to develop some language about smooth functions, which will occupy (1.31) through (1.33).

Notation 1.31 Let M be a manifold, $X \in \Gamma(M, T_*M)$ a vector field, and $f : M \to \mathbf{R}$ a smooth function. We will write $X(f)$ or just Xf for the smooth function

$$\begin{array}{ccc} M & \to & \mathbf{R} \\ m & \mapsto & X(m)(df(m)) \end{array}$$

(This is the same notation we introduced in (6.16.2).)

Inductive Definition 1.32. Let M be a manifold, and $f : M \to \mathbf{R}$ a smooth function.

i) We say that f *vanishes to order* 1 *at* $m \in M$ if $f(m) = 0$.

ii) Let k be a non- negative integer. We say that f *vanishes to order* k *at* $m \in M$ if, for every vector field $X \in \Gamma(M, T_*M)$, the function Xf vanishes to order $k-1$ at m.

Proposition 1.33. Let U be a parallelizable open set containing m and let $\{X_1, \ldots, X_n\}$ be a global basis for U. Let Y be any vector field and f any smooth function. Then if $X_i f(m) = 0$ for all i, we have $Yf(m) = 0$.

Proof. To compute $Yf(m)$ we can first restrict f and Y to U; thus we can assume Y is of the form $\sum_{i=1}^n g_i X_i$ for some smooth functions g_i. Thus

$$Yf(m) = \sum_{i=1}^n g_i(m) X_i f(m) = 0$$

Corollary 1.33.1. Under the assumptions of (1.33), suppose that $X_i f$ vanishes to order $k-1$ at m for all i. Then f vanishes to order k at m.

Exercise 1.33.2. Let $\phi : M \to N$ be a smooth map of manifolds and let $f : N \to \mathbf{R}$ be a smooth map that vanishes to order k at $\phi(m) \in N$. Show that $\phi \circ f$ vanishes to order k at m. (Use (II.4.12).)

Definition 1.34. Let $f, g : M \to \mathbf{R}$ be smooth functions. We say that f and g *agree up to order k at m* if the function $f - g$ vanishes to order k at m.

Remarks and Notation 1.35. Now that we know what it means for two *functions* to agree up to order k at m, we want to decide what it means for two *metrics* to agree up to order k at m.

We will offer a definition only in the case where $M = U$ is an open subset of a vector space V and $m = 0$ is the zero element in V.

For any metric $\mathbf{g}$ on U, and any two vector fields X, Y, let $< X, Y >_{\mathbf{g}}$ be the smooth function that is denoted $< X, Y >$ in (1.9).

Proposition 1.36. Let $\mathbf{g}$ and $\mathbf{h}$ be metrics on U. The following conditions are equivalent:

i There exists an isomorphism $\phi : V \to \mathbf{R}^n$ such that for every $j, k \in \{1, \ldots, n\}$, the functions

$$\left\langle \frac{\partial}{\partial x_j^\phi}, \frac{\partial}{\partial x_k^\phi} \right\rangle_{\mathbf{g}} \quad \text{and} \quad \left\langle \frac{\partial}{\partial x_j^\phi}, \frac{\partial}{\partial x_k^\phi} \right\rangle_{\mathbf{h}} \tag{1.36.1}$$

agree up to order k at 0.

ii For every isomorphism $\phi : V \to \mathbf{R}^n$ and for every $j, k \in \{1, \ldots, n\}$, the functions

$$\left\langle \frac{\partial}{\partial x_j^\phi}, \frac{\partial}{\partial x_k^\phi} \right\rangle_{\mathbf{g}} \quad \text{and} \quad \left\langle \frac{\partial}{\partial x_j^\phi}, \frac{\partial}{\partial x_k^\phi} \right\rangle_{\mathbf{h}} \tag{1.36.2}$$

agree up to order k at 0.

Definitions 1.37. 1) Two metrics $\mathbf{g}$ and $\mathbf{h}$ *agree up to order k at* 0 if either (and hence both) of the equivalent conditions in (1.36) are satisfied.

2) A metric $\mathbf{g}$ is *constant up to order k* if $\mathbf{g}$ and $F(G(\mathbf{g}))$ agree up to order k at 0.

3) Let (V, g_V) be a vector space with inner product, $(M, \mathbf{g}_M)$ a manifold with metric, and $f : V \to M$ a diffeomorphism. Then f is an *isometry up to order k* at 0 if the metrics $F(g_V)$ and $f^*\mathbf{g}_M$ agree up to order k at 0.

4) More generally, Let (V, g_V) be a vector space with inner product, U an open subset containing 0, $(M, \mathbf{g}_M)$ a manifold with metric, and $f : U \to M$ a diffeomorphism. Then f is an *isometry up to order* k at 0 if the metrics $F(g_V)|_U$ and $f^*\mathbf{g}_M$ agree up to order k at 0.

Exercise 1.37.1. Let $(V, \mathbf{g})$ be a vector space with a metric. Show that $\mathbf{g}$ is constant up to order k if and only if the identity map $f : (V, G(\mathbf{g})) \to (V, \mathbf{g})$ is an isometry up to order k.

Generalize to the case where f is defined on an open subset of V containing 0.

Proposition 1.38. If the inner products $\mathbf{g}_0$ and $\mathbf{h}_0$ are the same, then $\mathbf{g}$ and $\mathbf{h}$ agree up to order 1 at 0.

Proof. For any vector fields X and Y, we have

$$< X, Y >_{\mathbf{g}} (0) = \mathbf{g}_0(X, Y) = \mathbf{h}_0(X, Y) =< X, Y >_{\mathbf{h}} (0)$$

Corollary 1.38.1 Every metric is constant up to order 1.

Remark 1.38.2. In (5.9), we will give criteria under which a metric is constant up to order 2; these criteria will be important in justifying the use of special relativity as an approximation to general relativity. For use in (5.9), we record a criterion for being constant up to order 2:

Proposition 1.39. Let $\mathbf{g}$ be a metric on V, $\phi : V \to \mathbf{R}^n$ an isomorphism, and $X_i = \partial/\partial x_i^\phi$. Then the following are equivalent:

i)

$$X_i < X_j, X_k > (0) = 0 \qquad \text{for all } i, j, k$$

ii) $\mathbf{g}$ is constant up to order 2.

Proof. Let X be an arbitrary vector field. By (II.4.12), $X(0)$ is of the form $\sum_{i=1}^n \alpha_i X_i$ where the α_i are constants. So condition (1) implies that $X < X_j, X_k > (0) = 0$ for all i, j, as needed.

Remark 1.40. (1.39) remains true if V is replaced by an open subset $U \subset V$ containing 0; exactly the same proof works.

2. Covariant Derivatives

Scholium 2.1. Imagine a point moving along a parameterized curve γ in our manifold M. We can think of the parameter t as "time", so that $\gamma(t)$ represents the location of our particle at time t. It is natural to want to talk about the velocity and acceleration of such a particle.

It's easy to define velocity: The velocity at time t is $\gamma_*(t)$, the tangent vector to γ at $m = \gamma(t)$ (II.4.17). This velocity vector lives in the tangent space T_mM. At a later time t', when the particle is at m', the velocity vector lives in the tangent space $T_{m'}M$. That makes it difficult to define acceleration. Intuitively, acceleration should be the rate at which velocity is changing. But how can we compare two velocity vectors that live in entirely different tangent spaces?

In (II.6.11), we observed that although T_mM and $T_{m'}M$ are isomorphic as vector spaces, there is no *preferred* isomorphism between them and hence no preferred way to identify vectors in T_mM with vectors in $T_{m'}M$.

We observed also that if the n- dimensional manifold M is parallelizable, we can (by definition) choose a global basis (II.6.9) consisting of n everywhere linearly independent vector fields and use these to provide the missing identifications. Explicitly, if the vector fields are $X_1, \ldots, X_n$, we identify

$$\sum_{i=1}^{n} \alpha_i (X_i)_m \leftrightarrow \sum_{i=1}^{n} \alpha_i (X_i)_{m'} \tag{2.1.1}$$

.

But this is unsatisfactory for two reasons: First, M need not be parallelizable; that is, there might not exist a global basis. Second, even if M *is* parallelizable (or if we can replace M by some parallelizable open set containing both m and m'), there is no preferred choice of global basis and hence no preferred way of identifying tangent spaces.

We will address these concerns in reverse order. Thus we will temporarily dispense with the first concern by assuming that a global basis exists, and address the second concern: which of many global bases should we prefer?

In the absence of a metric, there is no clear answer. But if a metric has been specified, we can give preference to those global bases that are orthonormal (1.14). In fairly general circumstances, one can mimic the proof of (I.6.12.2) to show that such bases exist.

That narrows down the class of allowable bases, but it still fails to achieve uniqueness. To see how far we are from uniqueness, consider $\mathbf{R}^2$ with its usual metric and consider the following three orthonormal global bases:

$$\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$$

$$\{U, V\} = \left\{ \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right\}$$

$$\{X, Y\} \qquad \text{where } X \text{ and } Y \text{ are as in (1.16))}$$

Of these, the first and second both yield the same set of identifications among tangent spaces, but the third does not. So we need some way of distinguishing global bases that are "like" the first two from those that are "like" the third. It turns out that the Lie bracket (II.6.21) provides the appropriate distinction. We have

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] = 0 \qquad \text{and} \qquad [U, V] = 0$$

but according to (II.6.24)

$$[X, Y] \neq 0$$

Thus we confine our set of "preferred" global bases $\{X_i\}$ to those which are both orthonormal *and* such that

$$[X_i, X_j] = 0 \quad \text{for all} \quad i, j \tag{2.1.2}$$

It turns out that this is enough; any two orthonormal global bases satisfying (2.1.2) both yield the same identifications among tangent spaces. Thus we can solve the uniqueness problem if we can choose an orthonormal global basis satisfying (2.1.2).

Usually, however, we can't. First, our manifold might not be parallelizable; in other words, there might be no global bases at all. Second, even if there is a global basis, there might not be a global basis that is both orthonormal and satisfies (2.1.2).

Take, for example, the case of the 2- sphere. The 2-sphere has no global basis (II.6.8), so we can't even start this program. But suppose we restrict our attention to the submanifold $\Omega \subset \mathbf{S}^2$ considered in (1.19). On Ω, we found a global basis

$$\left\{ \frac{\partial}{\partial u^\phi}, \frac{\partial}{\partial v^\phi} \right\} \tag{2.1.3}$$

and this basis satisfies (2.1.2). But the calculation culminating in (1.19.2) shows that (2.1.3) is not orthonormal. (In other words, the matrix (1.19.2) is not of the form (1.13.2).) Conversely, the basis

$$\left\{ \frac{1}{\cos(v^\phi)} \frac{\partial}{\partial u^\phi}, \frac{\partial}{\partial v^\phi} \right\}$$

is orthonormal, but fails to satisfy (2.1.2). And in fact, Ω admits no global basis that satisfies both desiderata simultaneously. Nevertheless, we have learned (or at least asserted) something useful: orthonormality plus the vanishing of Lie brackets, when it is attainable, provides the uniqueness we are looking for. This will be a valuable lesson as we now proceed to the question of existence.

What, then, should we do if M is not parallelizable, so that no global basis exists? If m and m' are distinct points, how are we to identify the tangent spaces T_mM and $T_{m'}M$?

It's true that we don't have a global basis for M, but we can settle for less. Let C be an imbedded curve passing through both m and m'. Instead of looking for vector fields (i.e. maps from M to T_*M), we can look for maps $X_i : C \to T_*M$ which are sections in the sense that for all $c \in C$, $X(c) \in T_cM$. (In other words, the X_i are sections of the vector bundle you get when you restrict T_*M to C.) If we can choose the X_i to be linearly independent everywhere along C, we can use them to identify tangent spaces as in (2.1.1).

It turns out that such families of sections always exist. Of course, we need to worry about uniqueness. Here we take our cue from the lessons we learned in the case of global bases. First, we should require the X_i to be not just everywhere linearly independent but everywhere orthonormal; if they are evaluated at any $c \in C$ the result should be an orthonormal basis for T_cM.

Second, orthonormality is not enough. We also need a condition involving Lie brackets; it turns out that the precise statement of this condition requires us to think about choosing our preferred orthonormal families not just for one curve at a time but for all curves simultaneously; the Lie bracket condition will relate these choices to each other. After carefully sorting this out, we will be able to eliminate all ambiguity arising from the choice of the X_i.

Along the way, however, we've introduced a new source of ambiguity: the choice of the curve C that connects m with m'. It turns out that this ambiguity is fundamental; there is no natural way to resolve it. Indeed, we will be able to quantify the resulting ambiguity and use it as a measure of the *curvature* of M.

When we choose a curve C through m and m' and identify vectors in T_mM with their counterparts in $T_{m'}M$, we will say that we are *parallel- translating* vectors along C.

Parallel translation allows us to call two vectors in different locations "the same", and hence makes it possible to talk about the difference between two tangent vectors at two different points. This in turn makes it possible to talk about the rate at which a given vector field is "changing" as we move along the curve C. Therefore we will be able to define a sort of "directional derivative" for a vector field Y in the direction X, where X is a vector field having C as an integral curve. Conversely, if we knew how to define directional derivatives, we'd be able to define parallel translation: Given a vector $Y_m \in T_mM$, choose a vector field Y such that $Y(m) = Y_m$ and such that the directional derivative of Y along C is zero; then $Y_{m'} = Y(m') \in T_{m'}M$ gets identified with $Y_m \in T_mM$.

When it comes to making all of this precise, it turns out to be much easier to go from directional derivatives to parallel translation than the other way around. So we define directional derivatives (which are, in fact, called *covariant derivatives*) first and then use them to define parallel translation. Parallel translation will be perfectly well defined along any curve, but will depend on the curve chosen. Finally, we will investigate that dependence and use it to define curvature.

Temporarily Unmotivated Definition 2.2. Let X and Y be vector fields on M. For each open set $U \subset M$, define a map

$$\zeta(X,Y)_U : \Gamma(U, T_*M) \to \mathcal{C}(M)$$

by the formula

$$\begin{aligned} 2\zeta(X,Y)_U(Z) = X < Y, Z > +Y < X, Z > -Z < X, Y > \\ - < X, [Y,Z] > - < Y, [X,Z] > + < Z, [X,Y] > \end{aligned}$$

Here the action of the vector field X on the smooth real-valued function $< Y, Z >$ is as in (II.6.8.12).

Proposition 2.3. For $Z, W \in \Gamma(U, T_*M)$ and $\phi \in \mathcal{C}(U)$, we have

$$\zeta(X,Y)_U(\phi Z + W) = \phi\zeta(X,Y)_U(Z) + \zeta(X,Y)_U(W)$$

Proof. Compute, using (1.10.iv) and (II.6.23).

Proposition 2.4. Given vector fields X and Y, there exists a vector field $D_X Y$ such that for all open $U \subset M$ and all vector fields $Z : U \to T_*U$, we have

$$\zeta(X,Y)_U(Z) =< D_X Y, Z >$$

Proof. We apply Theorem (II.3.20) to the maps $\zeta_U = \zeta(X,Y)_U$. Hypothesis (II.3.20i) is just Proposition 2.3, and Hypothesis (II.3.20ii) is satisfied trivially, so the theorem is applicable and shows that there exists a one-form $\xi \in \Gamma(M,^* M)$ with

$$\zeta(X,Y)_U(Z) =< \xi, Z >$$

for all Z; now let $D_X Y$ be the vector field associated (1.9) to the one-form ξ.

Definition 2.5. The vector field $D_X Y$ is called the *covariant derivative of Y with respect to X*.

Proposition 2.6. For any vector fields X, Y and Z, we have

$$2 < D_X Y, Z > = X < Y, Z > + Y < X, Z > - Z < X, Y > \\ - < X, [Y, Z] > - < Y, [X, Z] > + < Z, [X, Y] >$$

Proof. This is an instantaneous consequence of Definitions (2.2) and (2.5).

Proposition 2.7. The covariant derivative satisfies the following properties:

i) $D_{\phi X_1 + X_2} Y = \phi D_{X_1} Y + D_{X_2} Y$

ii) $D_X(Y_1 + Y_2) = D_X(Y_1) + D_X(Y_2)$

iii) $D_X(\phi Y) = (X\phi)Y + \phi D_X Y$

iv) $X < Y, Z > = < D_X Y, Z > + < Y, D_X Z >$

v) $D_X Y - D_Y X = [X, Y]$

In these equations, as always, X, Y and Z are vector fields, ϕ is a smooth real-valued function, and the action of vector fields on smooth real-valued functions —in other words, the meaning of terms like $X(f)$—is given by (II.6.18.2).

Proof. Again, these are computations. Note that the proofs of (i) is essentially identical to the proof of Proposition (2.3).

Remark 2.7.1. Note that $D_{\phi X} Y = \phi D_X Y$ but in general $D_X(\phi Y) \neq \phi D_X Y$. It follows that X and Y enter into formula (2.2) in a way that is less symmetric than might appear to the casual observer.

Proposition 2.8. Let

$$E : \Gamma(M, T_* M) \times \Gamma(M, T_* M) \to \Gamma(M, T_* M)$$

be any function, and write $E_X Y$ for the image of (X, Y). Suppose that E satisfies

i) $E_{\phi X_1+X_2} Y = \phi E_{X_1} Y + E_{X_2} Y$

ii) $E_X(Y_1 + Y_2) = E_X(Y_1) + E_X(Y_2)$

iii) $E_X(\phi Y) = (X\phi)Y + \phi E_X Y$

iv) $X < Y, Z > = < E_X Y, Z > + < Y, E_X Z >$

v) $E_X Y - E_Y X = [X, Y]$

Then $E_X Y = D_X Y$ for all X and Y.

Proof. Apply (iv) three times to get

$$X < Y, Z > = < E_X Y, Z > + < Y, E_X Z > \tag{2.7.1}$$

$$Y < Z, X > = < E_Y Z, X > + < Z, E_Y X > \tag{2.7.2}$$

$$Z < X, Y > = < E_Z X, Y > + < X, E_Z Y > \tag{2.7.3}$$

Add (2.7.1) to (2.7.2) and subtract (2.7.3) to get

$$\begin{aligned} X < Y, Z > &+ Y < Z, X > - Z < X, Y > \\ &= < Z, E_X Y + E_Y X > + < Y, E_X Z - E_Z X > + < X, E_Y Z - E_Z Y > \\ &= 2 < Z, E_X Y > + < Z, E_Y X - E_X Y > + < Y, E_X Z - E_Z X > + < X, E_Y Z - E_Z Y > \end{aligned}$$

The three rightmost terms can be simplified using (iv), and the result follows from (2.6).

Motivation 2.8. We will think of the covariant derivative as describing the "rate of change of Y in the direction indicated by X". The contorted looking definition (2.2) is justified by the plausible looking properties (2.7) together with the fact (2.8) that no other definition could satisfy these properties.

Conditions (2.7i) through (2.7iii) simply say that the covariant derivative should behave like a directional derivative. The remaining two conditions are more substantive.

To gain some insight into (2.7iv), consider the case where $D_X Y = D_X Z = 0$, so that Y and Z are both unchanging in the X direction. Then the right side of (2.7iv) is clearly

0, so the left side is 0, which is to say that the inner product $< Y, Z >$ is also unchanging in the X direction. For example, if Y_m and Z_m are part of an orthonormal basis for $T_m M$, then $Y_{m'}$ and $Z_{m'}$ will be part of an orthonormal basis for $T_{m'} M$, provided m' can be reached from m by traveling along an integral curve for X.

Condition (2.7v) is a bit more mysterious; it is the “Lie bracket condition” referred to in Scholium (2.1) which is used to make the covariant derivative—and hence parallel translation—uniquely defined. Without (2.7v), the uniqueness theorem (2.8) would fail.

Remarks 2.9. In many treatments, a *covariant derivative* (or an *affine connection*) is defined to be any function $(X, Y) \mapsto D_X Y$ satisfying (2.7i) through (2.7iii); this definition does not require the manifold M to be equipped with a metric. In such treatments, D is then defined to be *compatible with the metric* if a metric is specified and (2.7iv) holds, and D is defined to be *torsion free* if (2.7v) holds. One then proves that there is a unique covariant derivative that is compatible with the metric and torsion free—in other words, one proves (2.8). This unique covariant derivative is called the *Levi-Civita connection*. In this book, we will have no need of covariant derivatives other than the Levi-Civita connection, so we have simply defined the Levi-Civita connection to *be* the covariant derivative.

Example 2.10. On $\mathbf{R}^n$ with one of the standard metrics (1.15), consider the vector fields

$$X_i = \frac{\partial}{\partial x_i}$$

Each $< X_j, X_k >$ is constant, so

$$X_i < X_j, X_k >= 0$$

for all i, j, and k.

Also,

$$[X_i, X_j] = 0$$

for all i and j by (II.6.21). Thus (2.6) reduces to

$$< D_{X_i} X_j, X_k >= 0$$

for all i, j and k.

Because the X_k are a global basis, it follows that for every vector field Y,

$$< D_{X_i} X_j, Y >= 0$$

Thus $< D_{X_i} X_j, - >=< 0, - >$, so, by the uniqueness in (1.10i),

$$D_{X_i} X_j = 0 \qquad \text{for all } i \text{ and } j \tag{2.10.1}$$

Exercise 2.11. Let x be the standard coordinate on $\mathbf{R} = \mathbf{R}^1$, let $X = \partial/\partial x$, let $g : \mathbf{R} \to \mathbf{R} - \{0\}$ be a differentiable function, and let $\mathbf{g}$ be the metric on $\mathbf{R}$ represented in terms of the global basis $\{X\}$ by the 1×1 matrix (g) (1.13).

When $\mathbf{R}$ is given the metric $\mathbf{g}$, show that

$$D_X X = \frac{g'}{2g} X$$

Example 2.12. Let $\Omega \subset \mathbf{S}^2$ be the coordinate patch described in (1.19) and consider the two vector fields

$$U = \frac{\partial}{\partial u^\phi} \qquad\qquad V = \frac{\partial}{\partial v^\phi}$$

We will compute the covariant derivative $D_U U$. Note first that $[U, U] = [U, V] = [V, U] = [V, V] = 0$ (II.6.21), so that we will always be able to ignore the final three terms in (2.6). Recall also from (1.19) that we have

$$< U, U >= \cos^2(v^\phi) \qquad < U, V >=< V, U >= 0 \qquad < V, V >= 1$$

Now, from (2.6),

$$\begin{aligned} 2 < D_U U, U > &= U < U, U >= \frac{\partial}{\partial u^\phi}(\cos^2(v^\phi)) = 0 \\ 2 < D_U U, V > &= 2U < U, V > -V < U, U >= -\frac{\partial}{\partial v^\phi}(\cos^2(v^\phi)) = 2\cos(v^\phi)\sin(v^\phi) \end{aligned} \tag{2.12.1}$$

Writing $D_U U = fU + gV$, (2.12.1) gives

$$< fU + gV, U >= f < U, U > +g < U, V >= f \cdot \cos^2(v^\phi) = 0$$

$$< fU + gV, V >= f < U, V > +g < V, V >= g = \cos(v^\phi)\sin(v^\phi)$$

so that

$$D_U U = \cos(v^\phi)\sin(v^\phi)V \tag{2.12.2}$$

Exercise 2.12.3 Show that

$$D_U V = D_V U = -\tan(v^\phi)U \tag{2.12.4}$$

$$D_V V = 0 \tag{2.12.5}$$

Remark 2.13. We still don't quite have everything we need. We want to define "directional derivatives" in the direction of a *tangent vector*, whereas (2.6) only tells us how to take directional derivatives in the direction of a *vector field*. Our next task is to remedy this situation.

Theorem 2.14. Given a vector field $Y \in \Gamma(M, T_*M)$, there exists a map of vector bundles

$$D(Y) : T_*M \to T_*M$$

such that for every vector field X and for every $m \in M$, we have

$$(D_X Y)(m) = D(Y)(X(m)) \in T_m M \tag{2.14.1}$$

Proof. For $U \subset M$ open, write write Y_U for the restriction of Y to U. For $X \in \Gamma(U, T_*M)$, define

$$\theta_U(X) = D_X(Y|_U) \in \Gamma(U, T_*M)$$

Claim 2.14.2 The maps θ_U constitute a sheaf map (II.2.32)

$$\theta : \widetilde{T_*M} \to \widetilde{T_*M}$$

Proof of (2.14.2). Condition (II.2.32i) is equivalent to (2.7i) and condition (II.2.32ii) is immediate from the definition.

Proof of (2.14) completed. Apply Theorem (II.2.35).

Definition 2.15. Let Y be a vector field on M, $m \in M$, and $t \in T_m M$. Then define the *covariant derivative of* Y *in the direction* t by the formula

$$D_t Y = D(Y)(t) \in T_m M$$

where $D(Y)$ is as in (2.14).

Proposition 2.16. The covariant derivative satisfies the following properties:

i) $D_{\alpha t_1 + t_2} Y = \alpha D_{t_1} Y + D_{t_2} Y$

ii) $D_t(Y_1 + Y_2) = D_t(Y_1) + D_t(Y_2)$

iii) $D_t(\phi Y) = t(d\phi_m) Y + \phi(m) D_t Y$

iv) $t < Y, Z >= \mathbf{g}_m(D_t Y, Z(m)) + \mathbf{g}_m(Y(m), D_t Z)$

v) If X is a vector field then

$$D_{X(m)} Y = D_X(Y)_m$$

Here t, t_1 and t_2 are tangent vectors, α is a real number, Y is a vector field, ϕ is a smooth real-valued function, $d\phi_m$ is as in (II.4.5.2), and $\mathbf{g}_m$ is as in (1.10ii).

Proof. These follow quickly from (2.7)(i-iv) and (2.13.1), except for (v), which is immediate from the definition.

Remark 2.17. The remainder of this section will be devoted to computing $D_t Y$ in an important special case.

Proposition 2.18. Let $I \subset \mathbf{R}$ be an open interval, let $\gamma : I \to M$ be an imbedded curve (II.4.17), let $v = \gamma_*(t)$ be the tangent vector to γ at a point m (II.4.17), and let Y be any vector field such that $Y \circ \gamma = 0$. (Equivalently, for every m' in the image of γ, $Y(m') = 0 \in T_{m'} M$.) Then

$$D_v Y = 0$$

Proof. Let Z be any vector field. Then the smooth function $< Y, Z >$ satisfies $< Y, Z > \circ \gamma = 0$, so by the definition of the tangent vector in (II.4.17), we have

$$v < Y, Z >= 0$$

Combining this with (2.15iv), we get

$$\begin{aligned} 0 &= \mathbf{g}_m(D_v Y, Z(m)) + \mathbf{g}_m(Y(m), D_v Z) \\ &= \mathbf{g}_m(D_v Y, Z(m)) + \mathbf{g}_m(0, D_v Z) \\ &= \mathbf{g}_m(D_v Y, Z(m)) \qquad \text{(because } \mathbf{g}_m \text{ is bilinear)} \end{aligned}$$

Now, at least after replacing M with an open subset of itself (which does not affect any of the equations), every element of $T_m M$ is of the form $Z(m)$ for some Z (II.2.6.1). Thus we have

$$\mathbf{g}_m(D_v Y, -) = 0$$

which, by nondegeneracy of $\mathbf{g}_m$, implies $D_v Y = 0$.

Corollary 2.18.1. Let $\gamma : I \to M$ be an imbedded curve, let $v = \gamma_*(m)$ be the tangent vector to γ at a point m, and let Y_1 and Y_2 be vector fields such that $Y_1 \circ \gamma = Y_2 \circ \gamma$. (Equivalently, Y_1 and Y_2 agree on the image of γ.) Then

$$D_v(Y_1) = D_v(Y_2)$$

Proof. We have

$$\begin{aligned} D_v(Y_1) - D_v(Y_2) &= D_v(Y_1 - Y_2) \qquad \text{(by (2.15ii))} \\ &= 0 \qquad \text{(by 2.18))} \end{aligned}$$

3. Parallel Translation

Now that we have defined the covariant derivative, we can define parallel translation along an imbedded curve.

3A. Parallelism

Definition 3.1. Let $I \subset \mathbf{R}$ be an open interval, and let $\gamma : I \to M$ be an imbedded curve. Let $v = \gamma_*(m)$ be the tangent vector to γ at a point m. Then a vector field Y is *parallel along* γ *at* m if

$$D_v Y = 0$$

Example 3.1.1. If $Y \circ \gamma = 0$, then Y is parallel along γ by (2.17.1).

Definition 3.2. Let $\gamma : I \to M$ be an imbedded curve. Then a vector field Y is *parallel along* γ if it is parallel along γ at m for every $m \in \gamma(I)$.

Proposition 3.3. Let X be a vector field and let γ be an imbedded curve that is an integral curve of X (II.6.13). Then a vector field Y is parallel along γ if and only if $D_X Y = 0$.

Proof. This is an immediate consequence of the definitions.

Example 3.4. A *vertical line* in $\mathbf{R}^2$ is an imbedded curve

$$\begin{array}{rccc} \gamma : & \mathbf{R} & \to & \mathbf{R}^2 \\ & t & \mapsto & (c, t) \end{array}$$

for some constant c.

We will find all the vector fields that are parallel along the vertical line γ.In the notation of Example (2.10), γ is an integral curve of the vector field X_2. Thus a vector field $Y = fX_1 + gX_2$ is parallel along γ if and only if

$$D_{X_2} Y = D_{X_2}(fX_1 + gX_2) = 0 \tag{3.4.1}$$

at every point in the image of γ.

Using (2.7ii), (2.7iii) and (2.10.1), we can rewrite (3.4.1):

$$\begin{aligned} 0 &= D_{X_2}(fX_1 + gX_2) \\ &= \frac{\partial f}{\partial x_2} X_1 + D_{X_2} X_1 + \frac{\partial g}{\partial x_2} X_2 + D_{X_2} X_2 \\ &= \frac{\partial f}{\partial x_2} X_1 + \frac{\partial g}{\partial x_2} X_2 \end{aligned}$$

Because X_1 and X_2 are a global basis, this can happen only if

$$\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_2} = 0$$

Thus a vector field $Y = fX_1 + gX_2$ is parallel along the vertical line γ if and only if $f \circ \gamma$ and $g \circ \gamma$ are constants.

Exercise 3.5. Let x be the standard coordinate on $\mathbf{R} = \mathbf{R}^1$, let $X = \partial/\partial x$, let $g : \mathbf{R} \to \mathbf{R} - \{0\}$ be a smooth function, and let $\mathbf{g}$ be the metric on $\mathbf{R}$ represented in terms of the global basis $\{X\}$ by the 1×1 matrix (g).

Let $\gamma : \mathbf{R} \to \mathbf{R}$ be the identity function, and let $f : \mathbf{R} \to \mathbf{R}$ be any smooth function.

When $\mathbf{R}$ is given the metric $\mathbf{g}$, show that the vector field fX is parallel along γ if and only if there exists a constant A such that

$$f = Ag^{-1/2}$$

(Use (2.11).)

Example 3.6 We will find all of the vector fields that are parallel along a line of longitude in $\mathbf{S}^2$.

We restrict our attention to the coordinate patch Ω of Example (1.19), and we use the notation of that example. We define a line of longitude to be the imbedded curve

$$\begin{array}{rccc} \gamma : & (-\frac{\pi}{2}, \frac{\pi}{2}) & \to & \Omega \\ & v & \mapsto & \phi^{-1}(u_0, v) \end{array} \tag{3.6.1}$$

for some constant u_0. According to (II.4.17.1), the tangent vector to γ is $V = \partial/\partial v^\phi$. So a vector field $Y = fU + gV$ is parallel along γ if

$$D_V Y = D_V(fU + gV) = 0 \tag{3.6.2}$$

at every point in the image of γ.

Using (2.15ii), (2.15iii), (2.12.4) and (2.11.5), we rewrite (3.6.2):

$$\begin{aligned} 0 &= \frac{\partial f}{\partial v^\phi}U + fD_V U + \frac{\partial g}{\partial v^\phi}V + gD_V V \\ &= \frac{\partial f}{\partial v^\phi}U - f\tan(v^\phi)U + \frac{\partial g}{\partial v^\phi}V \end{aligned} \tag{3.6.3}$$

Because U and V are everywhere linearly independent, (3.6.3) is equivalent to the two conditions

$$\frac{\partial f}{\partial v^\phi} = f \cdot \tan(v^\phi) \qquad\qquad \frac{\partial g}{\partial v^\phi} = 0$$

The solutions to these differential equations are

$$f = \frac{A}{\cos(v^\phi)} \qquad g = B$$

for constants A and B. Thus Y is parallel along γ if and only if there are constants A and B such that at every point in the image of γ we have

$$Y = \frac{A}{\cos(v^\phi)}U + BV$$

Specific Example 3.6.4. Taking $A = 0, B = 1$, we find that the vector field V is parallel along any line of longitude.

Specific Example 3.6.5. Taking $B = 0$, we find that any vector field

$$\frac{A}{\cos(v^\phi)}U$$

is parallel along any line of longitude.

Example 3.7. We will find all the vector fields that are parallel along a line of latitude in $\mathbf{S}^2$. Continue to use the notation of (1.19) and (3.6). We define a line of latitude to be an imbedded curve

$$\begin{array}{rccc} \gamma : & (-\pi, \pi) & \rightarrow & \Omega \\ & u & \mapsto & \phi^{-1}(u, v_0) \end{array} \tag{3.7.1}$$

for some constant v_0. The tangent vector to γ is $U = \partial/\partial u^\phi$. So a vector field $Y = fU + gV$ is parallel along γ if

$$D_U Y = D_U(fU + gV) = 0 \tag{3.7.2}$$

at every point in the image of γ.

Using (2.15ii), (2.15iii), (2.12.2) and (2.12.4), we rewrite (3.7.2):

$$\begin{aligned} 0 &= \frac{\partial f}{\partial u^\phi}U + fD_U U + \frac{\partial g}{\partial u^\phi}V + gD_U V \\ &= \frac{\partial f}{\partial u^\phi}U + f\cos(v^\phi)\sin(v^\phi)V + \frac{\partial g}{\partial u^\phi}V - g \cdot \tan(v^\phi)U \end{aligned} \tag{3.7.3}$$

Because U and V are everywhere linearly independent, (3.7.3) is equivalent to the two conditions

$$\frac{\partial f}{\partial u^\phi} = g \cdot \tan(v^\phi) \qquad\qquad f \cdot \cos(v^\phi)\sin(v^\phi) = -\frac{\partial g}{\partial u^\phi}$$

The solutions to these differential equations are

$$f = \frac{A\sin\Big(u^\phi \sin(v^\phi)\Big) - B\cos\Big(u^\phi \sin(v^\phi)\Big)}{\cos(v^\phi)} \tag{3.7.4}$$

$$g = A\cos\Big(u^\phi \sin(v^\phi)\Big) + B\sin\Big(u^\phi \sin(v^\phi)\Big) \tag{3.7.5}$$

for arbitrary constants A and B.

Therefore $Y = fU + gV$ is parallel along a line of latitude if and only if there are constants A and B such that (3.7.4) and (3.7.5) hold at every point in the image of γ.

(Note that on the image of γ, we have $v^\phi = v_0$, so that v^ϕ can be replaced by v_0 in conditions (3.7.4) and (3.7.5).)

Specific Example 3.7.6. Taking $A = 0$, $B = -\cos(v_0)$, equations (3.7.4) and (3.7.5) become

$$f = \cos\Big(u^\phi \sin(v_0)\Big)$$

$$g = \sin\Big(u^\phi \sin(v_0)\Big)$$

Thus the vector field

$$\cos\Big(u^\phi \sin(v_0)\Big)U - \cos(v_0)\sin\Big(u^\phi \sin(v^\phi)\Big)V \tag{3.7.6.1}$$

is parallel along a line of latitude.

Specific Example 3.7.7. Taking $A = 1$, $B = 0$, equations (3.7.4) and (3.7.5) become

$$f = \sin(\Big(u^\phi \sin(v_0)\Big) \; over \cos(v_0)$$

$$g = \cos\Big(u^\phi \sin(v_0)\Big)$$

Thus the vector field

$$\frac{\sin\Big(u^\phi \sin(v_0)\Big)}{\cos(v_0)}U + \cos\Big(u^\phi \sin(v_0)\Big)V \tag{3.7.7.1}$$

is parallel along a line of latitude.

Facts 3.8. Let $I \subset \mathbf{R}$ be an open interval containing the closed interval $[a, b]$. Let $\gamma : I \to M$ be an imbedded curve. Let $Y_{\gamma(a)} \in T_{\gamma(a)}M$ be a tangent vector. Then:

3.8.1. There exists an $\epsilon > 0$, an open subset $U \subset M$, and a vector field Y on U such that:

i) U contains the image under γ of the interval $(a - \epsilon, b + \epsilon)$.

ii) $Y(\gamma(a)) = Y_{\gamma(a)}$

iii) Y is parallel along γ.

3.8.2. If Y_1 and Y_2 are vector fields satisfying these conditions, then $Y_1 \circ \gamma = Y_2 \circ \gamma$.

3.8.3. To prove facts (3.8.1) and (3.8.2), one uses charts to reduce to the case $M = \mathbf{R}^n$ and then uses existence and uniqueness theorems from the theory of differential equations.

Definition 3.9. Given a curve γ and a tangent vector $Y_\gamma(a)$ as in (3.8), a vector field satisfying the conditions in (3.8.1) is called a *parallel continuation of Y along γ*.

Remark 3.10. The facts in (3.8) can be generalized to arbitrary (not necessarily imbedded) parameterized curves. For this purpose, one studies smooth functions $Y : I \to TM$ with the property that $Y(a) \in T_{\gamma(a)}M$ for all $a \in I$. Such a Y is called a *vector field along* γ; note in particular that this definition does not require $Y(a) = Y(a')$ even when $\gamma(a) = \gamma(a')$, so that when γ is not injective, Y cannot be thought of as an ordinary vector field composed with γ.

We will not need this additional level of generality, so we will not develop the theory of vector fields along a curve.

Definition 3.11. Henceforth, we will say that an imbedded curve γ *passes through* m if m is in the image of γ.

Definition 3.12. Let m and m' be points in M and let $\gamma : I \to M$ be an imbedded curve passing through m and m'. Write $m = \gamma(a)$, $m = \gamma(b)$.

Let $Y_m \in T_m M$ be a tangent vector. The *parallel translation of Y to m' along γ* is the vector

$$\tau^{\gamma}_{m,m'}(Y_m) = Y(m') \in T_{m'}M \tag{3.12.1}$$

where Y is a parallel continuation (3.9) of Y_m along γ. The vector $\tau^{\gamma}_{m,m'}(Y_m)$ is independent of the choice of Y by (3.8.2).

Proposition 3.13. The function $\tau^{\gamma}_{m,m'}$ (3.12.1) is an isomorphism of vector spaces.

Proof. We must show two things:

Claim 3.13.1 $\tau^{\gamma}_{m,m'}$ is a linear transformation.

Claim 3.13.2 $\tau^{\gamma}_{m,m'}$ is bijective.

Proof of 3.13.1. Given two vectors $Y_m, Z_m \in T_m M$ and a real number α, let Y and Z be parallel continuations of Y and Z along γ. Let U be the intersection of the domains of Y and Z and define a vector field $\alpha Y + Z$ on U by the formula

$$(\alpha Y + Z)(m) = \alpha Y(m) + Z(m)$$

Clearly $\alpha Y + Z$ is a parallel continuation of $\alpha Y(m) + Z(m)$ along γ. Thus

$$\begin{aligned}\tau^{\gamma}_{m,m'}(\alpha Y(m) + Z(m)) &= (\alpha Y + Z)(m') \\ &= \alpha Y(m') + Z(m') \\ &= \alpha \tau^{\gamma}_{m,m'}(Y_m) + \tau^{\gamma}_{m,m'}(Z_m)\end{aligned}$$

Proof of 3.13.2. $\tau^{\gamma}_{m',m}$ is an inverse for $\tau^{\gamma}_{m,m'}$.

Proposition 3.14. For $Y_m, Z_m \in T_m M$, we have

$$\mathbf{g}_{m'}(\tau^{\gamma}_{m,m'}(Y_m), \tau^{\gamma}_{m,m'}(Z_m)) = \mathbf{g}_m(Y_m, Z_m)$$

Proof. Let Y and Z be parallel continuations of Y_m and Z_m along γ. Let v be the tangent vector to γ at any point. Then by the definition of parallel continuation we have

$D_vY = D_vZ = 0$, so it follows from (2.16iv) that $v < Y, Z >= 0$. Combining this with the definiton in (II.4.17) and the formula in (II.4.15.2), we have

$$0 \quad = \quad v < Y, Z > \quad = \quad \frac{\partial}{\partial t} < Y, Z > (\gamma(t))$$

so that $< Y, Z > (\gamma(t))$ is constant as a function of t, and in particular

$$< Y, Z > (m') \quad = \quad < Y, Z > (m)$$

as advertised.

Corollary 3.15. Parallel translation takes orthonormal bases to orthonormal bases.

Remark 3.16. Given an orthonormal basis for T_mM, (3.15) says that parallel translation constructs an orthonormal basis at $T_{m'}M$, and (3.13.1) says that we can compute the parallel translation of any vector by linearity as suggested in (2.1.1). Thus we have fulfilled the program outlined in Scholium (2.1).

Example 3.17. We will compute the parallel translation map along a line of longitude on the 2-sphere. We continue to use the notation of Examples (1.19), (2.12), (3.6) and (3.7).

Let $m = \phi^{-1}(u_0, v_0)$ be a point in the coordinate patch Ω; think of m as the point with "longitude" u_0 and "latitude" v_0. Let γ be the imbedded curve (3.6.1) (the "line of longitude through m"), and let $m' = \phi^{-1}(u_0, v_0 + \Delta v)$ be another point in the image of γ. We will compute the map $\tau^{\gamma}_{m,m'}$.

Claim 3.17.1:

$$\tau^{\gamma}_{m,m'}\left(\frac{\partial}{\partial v^{\phi}}\right) = \frac{\partial}{\partial v^{\phi}}$$

(Note that the same symbol $\frac{\partial}{\partial v^{\phi}}$ is being used to denote tangent vectors in two distinct tangent spaces.)

Proof. By (3.6.1), the vector field V is parallel along any line of longitude.

Claim 3.17.2:

$$\tau^{\gamma}_{m,m'}\left(\frac{\partial}{\partial u^{\phi}}\right) = \frac{\cos(v_0)}{\cos(v_0 + \Delta v)}\frac{\partial}{\partial u^{\phi}}$$

Proof. By (3.6.2), the vector field

$$\frac{\cos(v_0)}{\cos(v^\phi)}U$$

is parallel along any line of longitude.

Conclusion 3.17.3. From (3.17.1), (3.17.2), and (3.13.1), we can compute the parallel translation of any tangent vector along our line of longitude:

$$\tau^\gamma_{m,m'}\left(\alpha\frac{\partial}{\partial u^\phi}+\beta\frac{\partial}{\partial v^\phi}\right)=\alpha\frac{\cos(v_0)}{\cos(v_0+\Delta v)}\frac{\partial}{\partial u^\phi}+\beta\frac{\partial}{\partial v^\phi}$$

Example 3.18. We will compute the parallel translation map along a line of latitude on the 2-sphere. We continue to use the notation of the preceding example.

Let $m=\phi^{-1}(0,v_0)$. Let γ be the imbedded curve (3.7.1) (the "line of latitude through m"), and let $m'=\phi^{-1}(u,v_0)$ be another point in the image of γ. We will compute the map $\tau^\gamma_{m,m'}$.

Claim 3.18.1:

$$\tau^\gamma_{m,m'}\left(\frac{\partial}{\partial u^\phi}\right)=\cos\Big(u\sin(v_0)\Big)\frac{\partial}{\partial u^\phi}-\cos(v_0)\sin\Big(u\sin(v_0)\Big)\frac{\partial}{\partial v^\phi} \qquad (3.18.1.1)$$

Proof. By (3.7.6.1), the vector field

$$\cos\Big(u^\phi\sin(v_0)\Big)U-\cos(v_0)\sin\Big(u^\phi\sin(v_0)\Big)V$$

is parallel along a line of latitude. Evaluating at $u^\phi=0$ and at $u^\phi=u$, we get the left and right sides of (3.18.1.1).

Claim 3.18.2:

$$\tau^\gamma_{m,m'}\left(\frac{\partial}{\partial v^\phi}\right)=\frac{\sin\Big(u\sin(v_0)\Big)}{\cos(v_0)}\frac{\partial}{\partial u^\phi}+\cos\Big(u\sin(v_0)\Big)\frac{\partial}{\partial v^\phi} \qquad (3.18.2.1)$$

Proof. By (3.7.7.1), the vector field

$$\frac{\sin\Big(u^\phi\sin(v_0)\Big)}{\cos(v_0)}U+\cos\Big((u^\phi)\sin(v_0)\Big)V$$

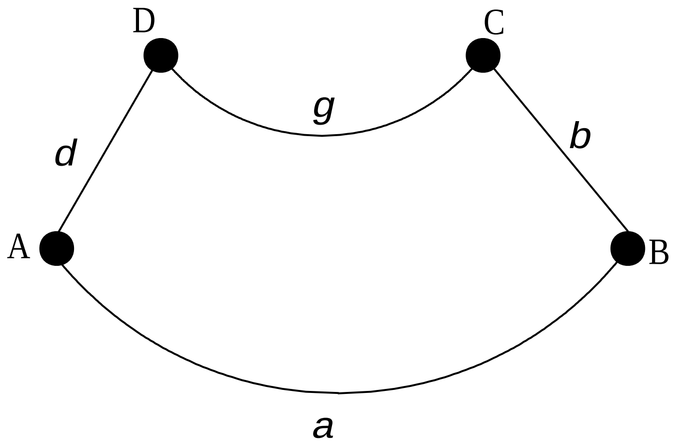
is parallel along a line of latitude. Evaluating at $u^\phi = 0$ and $u^\phi = u$, we get the left and right sides of (3.18.2.1).

Conclusion 3.18.3. From (3.18.1), (3.18.2), and (3.13.1), we have enough information to compute the parallel translation of any tangent vector along our line of latitude.

Exercise 3.19. Continue to use the notation of (3.17) and (3.19). Let $\Delta u, \Delta v \in (0, \pi/2) \subset \mathbf{R}$. Let

$$A = \phi^{-1}(0,0) \qquad B = \phi^{-1}(\Delta u, 0)$$
$$C = \phi^{-1}(0, \Delta v) \qquad D = \phi^{-1}(\Delta u, \Delta v)$$

Let α be a line of latitude through A and B, let β be a line of longitude through B and C, let γ be a line of longitude through A and D, and let δ be a line of latitude through D and C. (See Figure (3.19.1).)



(3.19.1)

Show that the diagram

$$\begin{array}{ccc} T_A M & \xrightarrow{\tau^{\alpha}_{A,B}} & T_B M \\ {\scriptstyle \tau^{\gamma}_{A,D}} \downarrow & & \downarrow {\scriptstyle \tau^{\beta}_{B,C}} \\ T_D M & \xrightarrow{\tau^{\delta}_{D,C}} & T_C M \end{array} \tag{3.19.2}$$

is *not* commutative.

3B. The Covariant Derivative as a Derivative.

In this subsection, we will clarify the sense in which the covariant derivative behaves like an ordinary directional derivative. First, we define the (ordinary) derivative of a map to a vector space.

Definition 3.20. Let V be a vector space, $I \subset \mathbf{R}$ an interval, and $\phi : I \to V$ a smooth function. Then for $t_0 \in I$ we define

$$\phi'(t_0) = \lim_{h \to 0} \frac{\phi(t_0 + h) - \phi(t_0)}{h} \tag{3.20.1}$$

To make sense of the limit, choose an isomorphism $V \approx \mathbf{R}^n$ and take limits in $\mathbf{R}^n$. It is easy to check that the value of (3.20.1) does not depend on the choice of isomorphism.

Remark 3.20.2. For $I \subset \mathbf{R}^m$ and a smooth map $\phi : I \to V$, the obvious generalization of (3.20) allows us to define the partial derivatives of ϕ with respect to the coordinates on $\mathbf{R}^m$.

Remarks 3.21. We want to define the derivative of a vector field along a curve. The problem is that a vector field is not a map to a single vector space; it's a map that takes each of its values in a *different* vector space. The idea is identify all these vector spaces with each other via parallel translation and then differentiate using (3.20). The result of that process turns out to be the covariant derivative in the direction of the tangent vector to the curve. Here is the precise result:

3.22. Proposition. Let $\alpha : I \to M$ be an imbedded curve, let Z be a vector field, and let $\mathbf{A} = \alpha(t_0)$ be a point on the image of α. Define

$$\hat{Z} : I \to T_m M$$

by

$$\hat{Z}(t) = \tau^{\alpha}_{\alpha(t),\alpha(t_0)}(Z(\alpha(t)))$$

Let $X = \alpha_*(t_0)$ be the tangent vector to α at $\mathbf{A} = \alpha(t_0)$. Then

$$\hat{Z}'(t_0) = D_X Z(m)$$

Proof. Choose an orthonormal basis $x_1, \ldots, x_n$ for $T_{\mathbf{A}}M$. By (3.8), we can assume (after replacing M with an open set U containing $\mathbf{A}$) there are vector fields X_i that are parallel along γ and satisfy $X_i(\mathbf{A}) = x_i$.

For any $\mathbf{B}$ in the image of γ, we know by (3.15) that the $X_i(\mathbf{B})$ form a basis for $T_{\mathbf{B}}M$. Therefore we can write

$$Z(\mathbf{B}) = \sum_{j=1}^{n} z_j(\mathbf{B}) X_j(\mathbf{B}) \tag{3.22.1}$$

for some real-valued functions z_j.

Now for fixed h, let P_h be the vector field

$$P_h = \sum_{j=1}^{n} z_j(\alpha(t_0 + h)) X_j$$

(Note that the coefficients $z_j(\alpha(t_0 + h))$ are constant functions.) Then P_h is parallel along α and agrees with Z at $\alpha(t_0 + h)$, so

$$\hat{Z}(t_0 + h) = P_h(\alpha(t_0))$$

Therefore

$$\begin{aligned}
\hat{Z}'(t_0) &= \lim_{h\to 0} \frac{P_h(\alpha(t_0)) - Z(\alpha(t_0))}{h} \\
&= \lim_{h\to 0} \frac{\sum_{j=1}^{n} z_j(\alpha(t_0 + h))x_j - \sum_{j=1}^{n} z_j(t_0)x_j}{h} \\
&= \sum_{j=1}^{n} \frac{\partial(z_j \circ \alpha)}{\partial t}(t_0)x_j \\
&= D_v Z(\mathbf{A})
\end{aligned}$$

where the final equality follows from the rules in (2.16) applied to (3.22.1), along with the observation that $D_v X_j = 0$ for all j (because X_j is parallel along α).

Corollary 3.22.2. Let X and Z be vector fields, let α be a integral curve of the vector field X, let $\mathbf{A} = \alpha(t)$ and let $\mathbf{B} = \alpha(t + x)$. Then up to terms of order ≥ 2 in x, we have

$$\tau^{\alpha}_{\mathbf{A},\mathbf{B}} Z(\mathbf{A}) \approx Z(\mathbf{B}) - x \cdot D_X Z(\mathbf{B})$$

Proof. By (3.22) we have (up to terms of order ≥ 2 in x)

$$\tau^{\alpha}_{\mathbf{B},\mathbf{A}} Z(\mathbf{B}) \approx Z(\mathbf{A}) + D_X Z(\mathbf{A})$$

Applying $\tau^{\alpha}_{\mathbf{A},\mathbf{B}}$ to both sides gives

$$\tau^{\alpha}_{\mathbf{A},\mathbf{B}} Z(\mathbf{A}) \approx Z(\mathbf{B}) - x \cdot \tau^{\alpha}_{\mathbf{A},\mathbf{B}} D_X Z(\mathbf{A}) \tag{3.22.2.1}$$

Now applying (3.22) again with $D_X Z$ in place of Z, we get

$$x \cdot \tau^{\alpha}_{\mathbf{A},\mathbf{B}} D_X Z(\mathbf{A}) \approx x \cdot D_X Z(\mathbf{B}) \tag{3.22.2.2}$$

Plugging (3.22.2.2) in to (3.22.2.1), the proposition follows.

Remarks 3.23. Proposition (3.22) bolsters the intuition that the covariant derivative measures the velocity of a vector field in a given direction. At some level, this is entirely circular: A velocity is a rate of change; to measure change, we have to know when two vectors are the same; to declare two vectors the same, we define parallel translation; to define parallel translation we use—the covariant derivative!

But (3.22) at least confirms tht our intuitions are internally consistent: *If* we think of the covariant derivative as a velocity, then we can think of parallel translated vectors as "the same"; and *if* we think of parallel translated vectors as "the same" then we can think of the covariant derivative as a velocity.

There remains the overwhelmingly important fact that the phrase "the same" changes its meaning when we parallel translate along a different curve. In Section 4, we will introduce the Riemann curvature tensor to keep track of this variation.

4. Riemann Curvature

4.1. Overview. Refer back to Figure (3.19.1), which shows a piece of the 2- sphere. In Exercise (3.19) , we calculated the results of parallel translation from A to C along two different paths and found that the results were different; in other words, diagram (3.19.2) does not commute.

Here's another way to say the same thing: Start with a tangent vector at A and parallel translate it all the way around the diagram from A to B to C to D and back to A again. The result is *not* in general the same vector you started with. To see that the two statements are equivalent, translate them into symbols. Noncommutativity of (3.19.2) means that

$$\tau^{\beta}_{B,C} \circ \tau^{\alpha}_{A,B} \neq \tau^{\delta}_{D,C} \circ \tau^{\gamma}_{A,D} \tag{4.1.1}$$

The fact that traveling all the way around the diagram fails to get you back to where you started means that

$$\tau^{\gamma}_{D,A} \circ \tau^{\delta}_{C,D} \circ \tau^{\beta}_{B,C} \circ \tau^{\alpha}_{A,B} \neq 1_{T_A M} \tag{4.1.2}$$

To get from (4.1.1) to (4.1.2), compose both sides on the left with the isomorphism $\tau^{\delta}_{C,D} \circ \tau^{\gamma}_{D,A}$and use (3.12.2); to get from (4.1.2) to (4.1.1) do the same thing with the isomorphism $\tau^{\delta}_{D,C} \circ \tau^{\gamma}_{A,D}$.

Intuitively, the noncommutativity of diagram (3.19.2) reflects the curvature of the 2-sphere; a vector p in $T_A M$ gets turned one way as it moves around the diagram in one direction and turned another way as it moves around the diagram in the other direction. By contrast, when S^2 is replaced by $\mathbf{R}^n$ (which is intuitively "flat"), the analogous diagram *is* commutative. (See (4.2.2).)

Our goal is to convert this insight into a *definition* of curvature. The idea is pretty simple: We parallel translate a vector all the way around a square, see how much the result differs from the identity, and take the difference as a measure of curvature. In Subsection 4A we will compute the parallel translation; in subsection 4B we will formalize the definition of curvature.

4A. Noncommutativity of parallel translation

Setup 4.2. We want to study parallel translation as we move around the sides of a small "rectangle" in M.

The easiest way to create a rectangle is to introduce coordinates. So let (U, ϕ) be a chart on M such that $\phi(U)$ contains the origin in $\mathbf{R}^n$, let x and y be small positive real numbers, and let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ be points such that

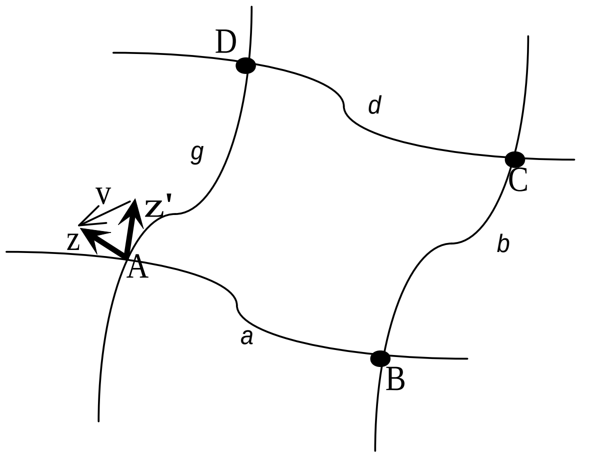
$$\begin{array}{ll} \phi(\mathbf{A}) = (0,0,0,\ldots,0) & \phi(\mathbf{B}) = (x,0,0,\ldots,0) \\ \phi(\mathbf{C}) = (x,y,0,\ldots,0) & \phi(\mathbf{D}) = (0,y,0,\ldots,0) \end{array}$$

Let $I, J \subset \mathbf{R}$ be open intervals containing $[0, x]$ and $[0, y]$ respectively and define four curves

$$\begin{array}{rccc} \alpha: & I & \to & M \\ & t & \mapsto & \phi(0,t,0,\ldots,0) \end{array} \qquad \begin{array}{rccc} \gamma: & I & \to & M \\ & t & \mapsto & \phi(y,t,0,\ldots,0) \end{array}$$

$$\begin{array}{rccc} \beta: & J & \to & M \\ & t & \mapsto & \phi(x,t,0,\ldots,0) \end{array} \qquad \begin{array}{rccc} \delta: & J & \to & M \\ & t & \mapsto & \phi(0,t,0,\ldots,0) \end{array}$$

Thus, inside M, the picture looks like Figure (4.2.1).



When the vector z is translated around the four sides of the rectangle, the resulting vector is z'. The difference v = z-z' is a measure of the curvature of M near A.

(4.2.1)

Now let z be a tangent vector at $\mathbf{A}$. We can parallel translate z around the rectangle, from $\mathbf{A}$ to $\mathbf{B}$ along α, from $\mathbf{B}$ to $\mathbf{C}$ along β, from $\mathbf{C}$ to $\mathbf{D}$ along γ and from $\mathbf{D}$ back to $\mathbf{A}$ along δ. Call the resulting vector z' and set $v = z - z'$.

Exercise 4.2.2. Let $M = \mathbf{R}^n$, let ϕ be the identity map from $\mathbf{R}^n$ to itself, and let z be an arbitrary tangent vector to $\mathbf{R}^n$ at the origin. Show that $v = 0$; in other words, show that in this case the diagram analogous to (3.19.1) is commutative.

Motivation 4.2. The vector $v = z - z'$ is a measure of the curvature of M near the point $\mathbf{A}$. Note that v depends on the chart (U, ϕ), the initial vector z, and the small numbers x and y. Fixing the chart and the vector z, we can view v as a function

$$\begin{array}{rccc} v: & I \times J & \to & M \\ & \cap & & \\ & \mathbf{R}^2 & & \end{array}$$

If we want to measure curvature *at* $\mathbf{A}$ rather than *near* $\mathbf{A}$, we should compute partial derivatives of v with respect to x and y. (These partialderivatives are defined a la (3.20.2).

To this end, let X and Y be the vector fields

$$X = \frac{\partial}{\partial x_1^\phi} \qquad Y = \frac{\partial}{\partial x_2^\phi}$$

Then we have:

Proposition 4.3. Let Z be any vector field such that $Z(\mathbf{A}) = z$. Then

$$v(0,0) = 0 \tag{4.3.1}$$

$$\frac{\partial v}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0) = 0 \tag{4.3.2}$$

$$\frac{\partial^2 v}{\partial x^2}(0,0) = \frac{\partial^2 v}{\partial y^2}(0,0) = 0 \tag{4.3.3}$$

$$\frac{\partial v}{\partial x \partial y}(0,0) = \big(D_X D_Y Z - D_Y D_X Z\big)(\mathbf{A}) \tag{4.3.4}$$

Remark 4.3.5. Equation (4.3.4) is remarkable: The right-hand side appears to depend on the vector field Z but the left- hand side depends only on the single value $z = Z(\mathbf{A})$. Thus the right-hand side's dependence on Z must be illusory, though it takes some calculation to pierce through the illusion.

Proof of 4.3. It is obvious that $v(x,0) = v(0,y) = 0$ for all x and y, from which (4.3.1), (4.3.2) and (4.3.3) all follow.

To prove (4.3.4), begin by setting:

$$z_1 = \tau^\alpha_{\mathbf{A},\mathbf{B}}(z)$$

$$z_2 = \tau^\beta_{\mathbf{B},\mathbf{C}}(z_1)$$

$$z_3 = \tau^\gamma_{\mathbf{C},\mathbf{D}}(z_2)$$

so that

$$z' = \tau^\delta_{\mathbf{D},\mathbf{A}}(z_3)$$

We will make a series of approximations, applying (3.22.2) to the vector fields Z, $D_X Z$ and $D_Y Z$ and always throwing away terms of order > 1 in x or y.

First, we have

$$z_1 \approx Z(\mathbf{B}) - xD_X Z(\mathbf{B})$$

This gives

$$\begin{aligned} z_2 &= \tau^{\beta}_{\mathbf{B},\mathbf{C}}(z_1) \\ &\approx \tau^{\beta}_{\mathbf{B},\mathbf{C}} Z(\mathbf{B}) - x\tau^{\beta}_{\mathbf{B},\mathbf{C}} D_X Z(\mathbf{B}) \\ &\approx Z(\mathbf{C}) - yD_Y Z(\mathbf{C}) - xD_X Z(\mathbf{C}) + xyD_Y D_X Z(\mathbf{C}) \end{aligned}$$

Therefore:

$$\begin{aligned} z_3 &= \tau^{\gamma}_{\mathbf{C},\mathbf{D}}(z_2) \\ &\approx \tau^{\gamma}_{\mathbf{C},\mathbf{D}} Z(\mathbf{C}) - y\tau^{\gamma}_{\mathbf{C},\mathbf{D}} D_Y Z(\mathbf{C}) - x\tau^{\gamma}_{\mathbf{C},\mathbf{D}} D_X Z(\mathbf{C}) + xy\tau^{\gamma}_{\mathbf{C},\mathbf{D}} D_Y D_X(\mathbf{C}) \\ &\approx Z(\mathbf{D}) + xD_X Z(\mathbf{D}) - yD_Y Z(\mathbf{D}) - xyD_X D_Y Z(\mathbf{D}) - xD_X Z(\mathbf{D}) + xyD_Y D_X(\mathbf{D}) \\ &= Z(\mathbf{D}) - yD_Y Z(\mathbf{D}) - xyD_X D_Y Z(\mathbf{D}) + xyD_Y D_X(\mathbf{D}) \end{aligned}$$

and finally:

$$\begin{aligned} z' &= \tau^{\delta}_{\mathbf{D},\mathbf{A}}(z_3) \\ &\approx \tau^{\delta}_{\mathbf{D},\mathbf{A}} Z(\mathbf{D}) - y\tau^{\delta}_{\mathbf{D},\mathbf{A}} D_Y Z(\mathbf{D}) - xy\tau^{\delta}_{\mathbf{D},\mathbf{A}} D_X D_Y Z(\mathbf{D}) + xy\tau^{\delta}_{\mathbf{D},\mathbf{A}} D_Y D_X Z(\mathbf{D}) \\ &\approx Z(\mathbf{A}) + yD_Y Z(\mathbf{A}) - yD_Y Z(\mathbf{A}) - xyD_X D_Y Z(\mathbf{A}) + xyD_Y D_X Z(\mathbf{A}) \\ &= Z(\mathbf{A}) - xyD_X D_Y Z(\mathbf{A}) + xyD_Y D_X Z(\mathbf{A}) \\ &= z - xyD_X D_Y Z(\mathbf{A}) + xyD_Y D_X Z(\mathbf{A}) \end{aligned}$$

as required.

4B. The Riemann Curvature Tensor

Discussion 4.4. Proposition (4.3) suggests that we ought to define curvature, more or less, as the function that takes three vector fields X, Y and Z, to the vector field $D_X D_Y Z - D_Y D_X Z$. We want this map to arise from a map of vector bundles

$$T_*M \otimes T_*M \otimes T_*M \to T_*M$$

so it can be identified with a section of the bundle

$$T^*M \otimes T^*M \otimes T^*M \otimes T_*M$$

This doesn't quite work, because the assignment

$$(X, Y, Z) \mapsto D_X D_Y Z - D_Y D_X Z \tag{4.4.1}$$

is not linear in X and Y. To make it work, we need a correction term; we will soon see that the right correction term is $-D_{[X,Y]}$. In the particular situation of (4.2) and (4.3), the correction term vanishes by (II.6.20).

The remainder of this section will be devoted to turning (4.4.1) into a rigorous definition.

Definition 4.5. Let X, Y and Z be vector fields on M. We define a new vector field

$$R(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z \tag{4.5.1}$$

If X, Y and Z are vector fields defined on open subsets of M, we restrict all three vector fields to the intersection of their domains, and then use (4.5.1) to define a vector field $R(X, Y)Z$ on that intersection.

Exercise 4.6. Continue to use the notation of Example (2.12) and use the computations summarized in (2.12.2), (2.12.4) and (2.12.5) to show that:

$$\begin{array}{ll} R(U,U)U = R(V,V)U = 0 & R(U,V)U = -R(V,U)U = -\cos^2(v^\phi)V \\ R(U,U)V = R(V,V)V = 0 & R(U,V)V = -R(V,U)V = U \end{array}$$

Proposition 4.7. Let ϕ be a real- valued function. Then

$$R(\phi X_1 + X_2, Y)Z = \phi R(X_1, Y)Z + R(X_2, Y)Z \tag{4.7.1}$$

$$R(X, \phi Y_1 + Y_2)Z = \phi R(X, Y_1)Z + R(X, Y_2)Z \tag{4.7.2}$$

$$R(X, Y)(\phi Z_1 + Z_2) = \phi R(X, Y)Z_1 + R(X, Y)Z_2 \tag{4.7.3}$$

Proof. Compute, using (2.7) and (II.6.23) repeatedly.

Corollary 4.8. There is a unique vector bundle map

$$R : T_*M \otimes T_*M \otimes T_*M \to T_*M$$

such that for all vector fields X, Y and Z and for all $m \in M$, we have

$$\Big(R(X,Y)Z\Big)(m) = R(X(m), Y(m), Z(m)) \in T_m M$$

Proof. This is an application of (II.3.25).

Definition 4.9. Given $m \in M$ and tangent vectors $x, y, z \in T_m M$, define

$$R(x,y)z = R(x,y,z) \in T_m M$$

where the R on the right is as in (4.8).

Definition 4.10. Given $m \in M$, vector fields $X, Y \in \Gamma(M, T_* M)$ and a tangent vector $z \in T_m M$, we write

$$R(X,Y)z = R(X(m), Y(m), z)$$

Proposition 4.11. There is a unique section

$$R \in \Gamma(M, Hom(T_* M \otimes T_* M \otimes T_* M, T_* M))$$

such that for all $m \in M$ and all $x, y, z \in T_m M$,

$$R(m)(x \otimes y \otimes z) = R(x,y)z$$

Proof. Use (II.3.27).

Definition 4.12. According to (II.3.30) and , there are natural isomorphisms of vector bundles

$$\begin{aligned} Hom(T_* M \otimes T_* M \otimes T_* M, T_* M) &\approx Hom(T_* M, \otimes T_* M \otimes T_* M, Hom(T^* M, \mathbf{R})) \\ &\approx Hom(T_* M \otimes T_* M \otimes T_* M \otimes T^* M, \mathbf{R}) \\ &= (T_* M \otimes T_* M \otimes T_* M \otimes T^* M)^* \\ &\approx T^* M \otimes T^* M \otimes T^* M \otimes T_* M \end{aligned}$$

Composing the section R of (4.11) with these natural isomorphisms, we get a section of $T^* M \otimes T^* M \otimes T^* M \otimes T_* M$ called the *Riemann curvature tensor.*

We will abuse both notation and language by using the symbol R and the name "Riemann curvature tensor" to denote the tensor just defined, the section R of (4.11) and the maps of (4.10), (4.9), (4.8) and (4.5).

Exercise 4.12.1. Continuing from Exercise 4.6, show that the Riemann curvature tensor for the 2-sphere (or, more precisely, for the coordinate patch (1.20.1) on the 2-sphere) is

$$\begin{aligned} R = &- \cos^2(v^\phi) du^\phi \otimes dv^\phi \otimes du^\phi \otimes \frac{\partial}{\partial v^\phi} \\ &+ \cos^2(v^\phi) dv^\phi \otimes du^\phi \otimes du^\phi \otimes \frac{\partial}{\partial v^\phi} \\ &+ du^\phi \otimes dv^\phi \otimes dv^\phi \otimes \frac{\partial}{\partial u^\phi} \\ &- dv^\phi \otimes du^\phi \otimes dv^\phi \otimes \frac{\partial}{\partial u^\phi} \end{aligned}$$

Exercise 4.12.2. Let $M = (\mathbf{R}^n, \mathbf{s})$ where $\mathbf{s}$ is a standard metric (1.15). Show that the Riemann curvature tensor vanishes.

Remarks 4.13. The vanishing of the Riemann curvature tensor is related to the uniqueness of parallel translation. Specificially, let m and m' be points, let $\gamma : I \to M$ and $\delta : I \to M$ be imbedded curves through m and m', and let $Y \in T_m M$ be a tangent vector. We seek conditions under which we can conclude that

$$\tau^\gamma_{m,m'}(Y) = \tau^\delta_{m,m'}(Y) \tag{4.13.1}$$

It is possible to prove that a sufficient set of conditions for (4.13.1) is:

i) The Riemann curvature tensor is identically zero

and *ii*) There exists a map

$$H : I \times I \to M$$

such that $H(t, 0) = \gamma(t)$, $H(t, 1) = \delta(s)$, $H(0, s) = m$ and $H(1, s) = m'$ for all $s, t \in I$.

A map H as in (ii) is called a *homotopy* from γ to δ. Such a homotopy always exists if M is diffeomorphic to $\mathbf{R}^n$. More generally, manifolds in which homotopies always exist are called *simply connected*. If M is simply connected, then, i) is a sufficient condition for (4.13.1).

5. Geodesics and the Exponential Map

5A. Geodesics

In (3.23) we learned to think of the covariant derivative as a measure of velocity. Now, after a few preliminaries, we will construct a measure of acceleration. We will then define a geodesic to be a curve along which the acceleration is zero.

Fact and Definition 5.1. Let $\gamma : I \to M$ be an imbedded curve. Then there exists a vector field V on M such that for each t, $V(\gamma(t)) = \gamma_*(t)$. Any such vector field is called a *velocity field* for γ.

Proposition 5.2. Let γ be an imbedded curve, let V and W be velocity fields for γ. Let $v = V(m) = \gamma_*(m)$. Then

$$D_v V = D_v W$$

Proof. Apply (2.18.1).

Definition 5.3. Let γ be an imbedded curve. The *acceleration vector* to γ at a point $m = \gamma(t)$ is the vector $D_v V$, where $v = \gamma_*(m)$ and V is a velocity field for γ. This is well-defined (i.e. independent of the choice of velocity field) by (5.2).

Definition 5.4. An imbedded curve γ is a *geodesic* if at every point in the image of γ, the acceleration vector is zero.

Exercise 5.4.1. Let $\phi : M \to N$ be an isometry, and let $\gamma : I \to M$ be a parameterized curve. Show that if $\phi \circ \gamma$ is a geodesic in N then γ is a geodesic in M.

Proposition 5.5. Let V be a velocity field for γ. Then γ is a geodesic if and only if V is parallel along γ.

Remark 5.6. Definition (5.4) can be generalized to curves that are not imbedded. Suppose the interval $I \subset \mathbf{R}$ can be covered by subintervals I_α such that for every α, the restriction $\gamma|_{I_\alpha}$ to I_α is an imbedded curve. Then we call γ a *geodesic* if each $\gamma|_{I_\alpha}$ is a geodesic.

Exercises 5.7.

i) Endow $\mathbf{R}^n$ with a standard metric (1.13). Let $\gamma : I \to \mathbf{R}^n$ be a parameterized curve. Show that γ is a geodesic if and only if it has the form

$$\begin{array}{rccc} \gamma: & I & \to & \mathbf{R}^n \\ & t & \mapsto & (t\alpha_1 + \beta_1, \ldots, t\alpha_n + \beta_n) \end{array} \qquad \textit{nudge} \tag{5.7.1}$$

for some $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbf{R}^n$.

ii) More generally, let V be any vector space with a constant metric (1.24), or even more generally let U be an open subset of V containing 0 (again with a constant metric). Show that the result of (i) continues to hold.

iii) Give $\mathbf{R}$ the metric described in (2.11). Show that $\gamma : I \to \mathbf{R}$ is a geodesic if and only if it satisfies a differential equation

$$\gamma'(x) = Ag(\gamma(x))^{-1/2}$$

for some constant A. (Use (3.5).)

iv) On the 2-sphere, show that every line of longitude is a geodesic. By rotating the coordinate system, show that every great circle is a geodesic. Show that every geodesic is a segement of a great circle.

Remarks 5.8. Let $(V, \mathbf{g})$ be a vector space with a metric. Exercise (5.7ii) says that if $\mathbf{g}$ is constant, then all straight lines, and in particular all straight lines through the origin, are geodesics. It's natural to ask about a sort of converse: Suppose all straight lines through the origin are geodesics. How close is $\mathbf{g}$ to being constant? Proposition (5.9) answers this question.

Proposition 5.9. Let V be a vector space with metric $\mathbf{g}$. Suppose that each "straight line through the origin"

$$\begin{array}{rccc} \gamma: & I & \to & \mathbf{R}^n \\ & t & \mapsto & tv \end{array} \qquad v \in V \text{ fixed} \tag{5.9.1}$$

is a geodesic. Then **g** is constant up to order 2 (1.35).

Proof. Let $\phi : V \to \mathbf{R}^n$ be any isomorphism and let X_i be the vector field $\partial/\partial x_i^{\phi}$.

Given two functions f and g defined on V, we will write $f \equiv g$ to mean that $f(0) = g(0)$. Then by (1.39), it suffices to show that

$$X_i < X_j, X_k > \equiv 0 \qquad \text{for all } i \text{ and } j \tag{5.9.2}$$

To prove (5.9.2), we will need:

Claim 5.9.3. Let

$$\phi(v) = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$$

Then at any point $m \in V$, the tangent vector to (5.9.1) is given by

$$\gamma_*(m) = \sum_{i=1}^{n} \alpha_i X_i(m)$$

Accepting this claim for the moment, and using the fact that (5.9.1) is a geodesic, it follows that if we set

$$X = \sum_{i=0}^{n} \alpha_i X_i$$

then

$$D_X X \equiv 0 \tag{5.9.4}$$

As special cases of (5.9.4), we have

$$D_{X_i} X_i \equiv 0 \qquad \text{for all } i \tag{5.9.5}$$

and

$$D_{X_i+X_j}(X_i + X_j) \equiv 0 \qquad \text{for all } i \text{ and } j \tag{5.9.6}$$

Now we can compute

$$\begin{aligned} 0 &\equiv D_{X_i+X_j}(X_i + X_j) && \text{(by (5.9.6)} \\ &\equiv D_{X_i}X_i + D_{X_i}X_j + D_{X_j}X_i + D_{X_j}X_j && \text{(by (2.7i) and (2.7(ii)} \\ &\equiv D_{X_i}X_j + D_{X_j}X_i && \text{(by 5.9.5)} \\ &\equiv 2D_{X_i}X_j && \text{by (2.7.5) and (II.6.20)} \end{aligned}$$

so that

$$D_{X_i}X_j \equiv 0 \qquad \text{for all } i \text{ and } j \tag{5.9.7}$$

Now use (2.7iv) and (5.9.7) to get

$$\begin{aligned} X_i < X_j, X_k > &\equiv < D_{X_i}X_j, X_k > + < X_j, D_{X_i}X_k > \\ &\equiv < 0, X_k > + < X_j, 0 > \\ &\equiv 0 \end{aligned}$$

proving (5.9.2) and thereby completing the proof of (5.9) as advertised.

It remains only to prove the claim:

5.9.8. Proof of (5.9.3). We have

$$\begin{aligned} (\phi \circ \gamma)(t) &= \phi(tv) \\ &= t\phi(v) \qquad \text{because } \phi \text{ is a linear transformation} \\ &= t(\alpha_1, \ldots, \alpha_n) \qquad (5.9.3.1) \end{aligned}$$

so, writing $m = \gamma(t)$, we have

$$\begin{aligned} \gamma_*(m) &= \gamma_{*t}\left(\frac{\partial}{\partial t}\right) && \text{by definition} \\ &= \phi_{*m}^{-1} \circ (\phi_{*m} \circ \gamma_{*t})\left(\frac{\partial}{\partial t}\right) \\ &= \phi_{*m}^{-1}\left((\phi \circ \gamma)_{*t}\left(\frac{\partial}{\partial t}\right)\right) && \text{by (II.4.16)} \\ &= \phi_{*m}^{-1}\left(\sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}\right) && \text{from (5.9.3.1)} \\ &= \sum_{i=1}^{n} \alpha_i \phi_{*m}^{-1}\left(\frac{\partial}{\partial x_i}\right) \\ &= \sum_{i=1}^{n} \alpha_i X_i(m) && \text{from (II.4.15.2)} \end{aligned}$$

as needed.

Remark 5.10. In view of Remark (1.40), the proof of (5.9) works equally well when V is not a vector space, but an open subset of a vector space containing the 0.

5B. Geodesic Deviation

Next we will devise a formal measure of the tendency for nearby geodesics to drift apart and show that it is completely controlled by the Riemann curvature tensor.

Definition 5.11. A *family of geodesics* consists of two open intervals $I, J \subset \mathbf{R}$ and a smooth function $\gamma : I \times J \to M$ such that for any fixed $s \in I$, the map

$$\begin{array}{rccc} \gamma(s, -): & J & \to & M \\ & t & \mapsto & \gamma(s,t) \end{array} \tag{5.11.1}$$

is a geodesic.

At any point $m \in \gamma(I \times J)$, define the *vector of geodesic deviation* to be the tangent vector to the imbedded curve

$$\begin{array}{rccc} \gamma(-, t): & I & \to & M \\ & s & \mapsto & \gamma(s,t) \end{array} \tag{5.11.2}$$

As t varies, think of the vector of geodesic deviation as a measure of how quickly the geodesics (5.11.1) spread apart.

Facts 5.12. Given a family of geodesics as in (5.11), there exist vector fields V and W such that V is a velocity vector field for all the geodesics (5.11.1) and W is a velocity vector field for all the curves (5.11.2).

Remark 5.12.1. We will want to measure the acceleration of the geodesic deviation as we move along a given geodesic (5.11.1). In other words, we want to measure $D_V D_V W$. The next proposition shows that the geodesic deviation is completely controlled by the curvature tensor.

Proposition 5.13. With the notation of (5.11) and (5.12), we have

$$D_V D_V W = R(V, W)V$$

where R is the Riemann curvature (4.4).

Proof. By (II.6.2), $[V, W] = 0$. Thus

$$R(V, W)V = D_V D_W V - D_W D_V V$$

$$
\begin{aligned}
&= D_V D_W V - D_W(0) && \text{because (5.11.1) is a geodesic} \\
&= D_V D_W V \\
&= D_V D_V W && \text{by (2.7v)}
\end{aligned}
$$

C. Ricci Curvature

The Ricci curvature tensor is a measure of curvature that includes some but not all of the information that is imbedded in the Riemann curvature tensor. The geometric significance of the Ricci tensor will appear in Chapter IV.

Definition 5.14. Given a vector space V, recall from (I.5.2.6) that we have a contraction map

$$
\begin{array}{ccc}
V \otimes V \otimes V \otimes V^* & \to & V \otimes V \\
u \otimes v \otimes w \otimes f & \mapsto & f(v)(u \otimes w)
\end{array}
$$

Applying this map separately to each $T_m M \subset T_* M$, we get a map of vector bundles

$$
T^*M \otimes T^*M \otimes T^*M \otimes T_*M \to T^*M \otimes T^*M \tag{5.14.1}
$$

(To check smoothness of (5.14.1), work with coordinate patches U that are parallelizable.)

Recall from (4.12) that the Riemann curvature tensor R is a section

$$
R : M \to T^*M \otimes T^*M \otimes T^*M \otimes T_*M \tag{5.14.2}
$$

Composing (5.14.2) with (5.14.1) gives a section

$$
\hat{R} \in \Gamma(M, T^*M \otimes T^*M)
$$

$\hat{R}$ is called the *Ricci curvature tensor*.

Exercise 5.15. Return yet again to the example of the 2-sphere, most recently revisited in Exercise (5.7iv). Show that the Ricci tensor is

$$
-\cos^2(v) du^\phi \otimes du^\phi - dv^\phi \otimes dv^\phi
$$

Definition 5.16. Let $\mathbf{g} : T_*M \to T^*M$ be the isomorphism of (1.8). $\mathbf{g}^{-1}$ induces an isomorphism of vector bundles.

$$\begin{array}{rccc} \mathbf{g}^{-1} \otimes 1 : & T^*M \otimes T^*M & \to & T_*M \otimes T^*M \\ & \xi \otimes \rho & \mapsto & \mathbf{g}^{-1}(\xi) \otimes \rho \end{array}$$

Now consider the smooth function

$$S : M \xrightarrow{\hat{R}} T^*M \otimes T^*M \xrightarrow{\mathbf{g}^{-1} \otimes 1} T_*M \otimes T^*M \xrightarrow{\text{Trace}} \mathbf{R}$$

where the trace map is given by $(f, v) \mapsto f(v)$ (see I.2.18). The function S is called the *scalar curvature* function on M.

Exercise 5.17. Continuing from Exercise (5.15), show that the scalar curvature function on the 2-sphere is the constant function with value -2.

Definition 5.18. The *Einstein tensor* is

$$\hat{R} - \frac{1}{2} S \mathbf{g}$$

where $\hat{R}$ is the Ricci curvature tensor (5.14), S is the scalar curvature function (5.15) and $\mathbf{g}$ is the metric. The Einstein tensor will play a major role in future chapters.

Exercises 5.18.1 Show that for $\mathbf{R}^n$, the Einstein tensor is zero. Show that for $\mathbf{S}^2$, the Einstein tensor is zero.

D. The Exponential Map

Scholium 5.19. In special relativity, observers travel through a four- dimensional vector space. In general relativity, observers travel through a four- dimensional manifold. (This will be made precise in Chapter IV.) Measurements of time and distance are simplest in special relativity, where the norm of a vector (I.6.8) can be thought of as a sort of "distance from the origin" (though this "distance" has different physical interpretations depending on whether $\mathbf{g}(v, v)$ is positive, negative or zero).

In general relativity, we'd like special relativistic measurements to continue making sense. We accomplish this by thinking of the vector space T_mM as a linear approximation to the manifold M. This requires a map from T_mM (or at least a part of T_mM) to M. The map we construct is called the *exponential map* (for reasons that will be indicated in Exercise 5.20) and denoted $\exp_m : T_mM \to M$.

Here's how we construct the exponential map: Given $v \in T_mM$, find a geodesic $\gamma : I \to M$ such that $\gamma(0) = m$ and $\gamma_*(0) = v$. Such a geodesic exists and is unique by the theory of ordinary differential equations.

(Here "uniqueness" means that if $\upsilon : J \to M$ is another geodesic with $\upsilon(0) = m$ and $\upsilon_*(0) = v$, then $\gamma|_{I\cap J} = \upsilon|_{I\cap J}$.) Thus we can set

Provisional Definition 5.19.1.

$$\exp_m(v) = \gamma(1) \in M$$

There are two problems with (5.19.1). First, the existence theorem does not guarantee that 1 is in the domain of γ. Thus $\exp_m(v)$ might be undefined. Second, there is nothing to guarantee that the map $\exp_m$ is one-one, which limits its usefulness as an "identification" between its domain and its image.

However, we have the following:

Facts 5.19.2. For $m \in M$, think of T_mM as a manifold via (II.1.3.5). Then there is an open set $U \in T_mM$ such that

i) $0 \in U$

ii) $\exp_m(v)$ is defined for every $v \in U$

iii) $\exp_m(U)$ is an open subset of M (II.1.4) and hence a manifold in its own right via (II.1.4.1).

iv) $\exp_m : U \to \exp(U)$ is a diffeomorphism.

Fact (i) is trivial (use the geodesic that is constant at m); Fact (ii) is another consequence of the theory of ordinary differential equations, and Facts (iii) and (iv) can be proved by invoking the Inverse Function Theorem from advanced calculus.

Remarks 5.19.3. In view of (5.15.2), we can upgrade (5.19.1) from a provisional definition to a full- fledged definition, provided we restrict the domain of $\exp_m$ to an open set U as in (5.19.2). Of course, this open set U is not unique, so it is a slight abuse of language to call $\exp_m$ *the* exponential map. In fact there is (for each m) a family of exponential maps, one for each domain U. But, exponential maps defined on U_1 and U_2 will agree on $U_1 \cap U_2$, so there is never any ambiguity about the expression $\exp_m(v)$.

Thus we define:

Definition 5.19.4. Given $m \in M$, an *exponential map* $\exp_m$ consists of an open set U as in (5.19.2) and a map $\exp_m : U \to M$ defined as in (5.19.1). We often abuse language by calling an exponential map *the* exponential map.

Remark 5.19.5. You might think we could remove all ambiguity by taking U to be the union of all open sets on which an exponential map can be defined, and then reserving the phrase *the* exponential map for the exponential map with domain U. In fact, U is open by (II.1.5d). But this doesn't work, because $\exp_m$ might not be one-to-one on such a large domain, and that spoils (5.19.2iv).The problem is that geodesics can cross. Imagine, for example, starting at a point m on the equator of the 2-sphere. A geodesic that sets out along the equator and a geodesic that starts out along a meridian of longitude will eventually meet on the opposite side of the sphere. Thus for appropriate choices of tangent vectors v_1 and v_2 pointing in these directions, we can have $\exp_m(v_1) = \exp_m(v_2)$.

Exercise 5.20. Let $M = \mathbf{R}$. Write x for the standard coordinate on $\mathbf{R}$ and $\partial/\partial x$ for the corresponding vector field. Let $\mathbf{g}$ be the usual metric on $\mathbf{R}$, so that

$$\mathbf{g}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 1$$

Show that for any $x \in M$, the exponential map is given by

$$\exp_x\left(\alpha\frac{\partial}{\partial x}\right) = x + \alpha \in \mathbf{R}$$

Exercise 5.21. Let $M = \mathbf{R} - \{0\}$. Write x for the standard coordinate on $\mathbf{R}$ and $\partial/\partial x$ for the corresponding vector field. Let $\mathbf{g}$ be the metric defined by

$$\mathbf{g}_x\left(\alpha\frac{\partial}{\partial x}, \beta\frac{\partial}{\partial x}\right) = \frac{\alpha\beta}{x^2}$$

(This metric arises naturally in the theory of Lie groups.) Show that

$$\exp_x\left(\alpha\frac{\partial}{\partial x}\right) = xe^{\alpha/x}$$

(Use (5.11iii).)

Remarks 5.22. Immediately from the definition, every manifold M is a union of open sets which are diffeomorphic to open sets in $\mathbf{R}^n$. We can ask for something stronger: Is it true that every manifold *with metric* $(M, \mathbf{g})$ is a union of open sets that are *isometric* to $(\mathbf{R}^n, \mathbf{s})$ where $\mathbf{s}$ is a standard metric?

The answer is no, but, using the exponential map, we will now show that another answer is "yes, at least up to second order":

Proposition 5.23. Let $(M, \mathbf{g})$ be a manifold with metric, and let $m \in M$ be any point. Let $U \subset T_mM$ be an open set on which the exponential map $\exp_m$ is defined and let $\Omega = \exp_m(U) \subset M$. Let $G(\mathbf{g}_m)$ be the metric on U defined in (1.29). Then the map

$$(U, G(\mathbf{g}_m)) \xrightarrow{\exp_m} (\Omega, \mathbf{g}|\Omega) \tag{5.23.1}$$

is an isometry up to order 2 (1.37).

Proof. Let $\mathbf{h} = \exp^*(\mathbf{g}|_\Omega)$. It follows immediately from (5.4.1) that straight lines are geodesics in $(U, \mathbf{h})$. Thus, by (5.9) and (5.10), $\mathbf{h}$ is constant up to order 2. This means (1.37(1)) that:

$$F(G(\mathbf{h})) \text{ agrees with } \mathbf{h} \text{ up to order 2 at 0}$$

Chasing through the construction (1.29), you can verify that $G(\mathbf{h}) = \mathbf{g}_m$, so we have:

$$F(\mathbf{g}_m) \text{ agrees with } \mathbf{h} \text{ up to order 2 at 0}$$

which (1.37(4)) says precisely that (5.23.1) is an isometry up to order 2.

Remarks 5.23.2. You can think of (5.23) and its proof as saying that near a given point m, $(M, \mathbf{g})$ is well approximated by the tangent space T_mM, endowed with the constant metric $F(\mathbf{g}_m)$. On the other hand, by (1.28), the tangent space is linearly isometric to $(\mathbf{R}^n, \mathbf{s})$ where $\mathbf{s}$ is a standard metric. Thus we can summarize our conclusion (in slightly imprecise language) by saying that any manifold with metric looks *locally* and *up to second order* like $\mathbf{R}^n$ with a standard metric.