Some New K-Theoretic Invariants for Commutative Rings

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In 1979, Spencer Bloch introduced a doubly indexed family of invariants, called the *higher Chow groups*, associated to a commutative ring, or more generally to an arbitrary scheme. In keeping with the spirit of the conference, I will use the language of ring theory rather than that of algebraic geometry. Thus we will be interested in a commutative ring R and its higher Chow groups $Ch^j(R,n)$.

Bloch conjectured that there was a close relationship between higher Chow groups and algebraic K-theory, and sketched a program for establishing such a relationship. In attempting to carry out that program, I was led to conjecture the existence of a spectral sequence

$$E_2^{p,q} = Ch^{-q}(R,(-p-q)) \Longrightarrow K_{-p-q}(R).$$

At around the same time, Beilinson formulated a conjectural framework for a doubly indexed *motivic cohomology theory* $H^\bullet(R,\bullet)$. According to Beilinson, there ought to be a spectral sequence, analogous to the topologists' Atiyah-Hirzebruch spectral sequence, converging from motivic cohomology to algebraic K-theory. It soon became apparent that the conjectures were intimately linked, and related by a new conjecture that the higher Chow groups *are* a motivic cohomology theory.

In the course of trying to establish the Atiyah-Hirzebruch spectral sequence for higher Chow theory, I needed a good patching theory for modules of finite projective dimension. Suppose that we are given a pullback square of commutative rings

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & R_{12} \end{array} .$$

Under certain circumstances, we know that there is a one-to-one correspondence

$$\left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{projective } R\text{-modules}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{patching data}\end{array}\right\}$$

Here *patching data* consist of a projective module P_1 over R_1, a projective module P_2 over R_2, and an isomorphism (the "patch") $\alpha : P_1 \otimes_{R_1} R_{12} \to P_2 \otimes_{R_2} R_{12}$.

(The classical theorem along these lines is in [Mi]. See [PAI] and [PB] for the most general formulations.)

What I needed was an analogous correspondence between isomorphism classes of *finite projective dimension* R-modules and patching data in which the P_i are permitted to be modules of arbitrary finite projective dimension. But unless one is willing to impose additional conditions, involving the vanishing of various Tors, no such correspondence is possible. (This was already clear from [PMFPD] and Man's thesis [Man].) These Tor conditions are far too restrictive for the applications to higher Chow theory. Thus I was led to reformulate the necessary patching condition in terms of *complexes* rather than modules.

The first task is to identify those complexes that act like objects of finite projective dimension. The right answer is the class of *perfect complexes*, or complexes that admit a quasi-isomorphism from a bounded complex of finitely generated projectives. The next task is to decide what we mean by a "patch". It turns out that isomorphisms of complexes are not a large enough family of patching maps; instead one must be willing to patch along arbitrary *quasi*-isomorphisms (a quasi-isomorphism is a map of complexes that induces an isomorphism on homology). It turns out that there is indeed a good patching theory along these lines, and that the switch from modules to complexes overcomes many technical difficulties.

Although the patching theorem is stated for complexes, it has immediate consequences for modules. Let P_1 and P_2 be modules of finite projective dimension over R_1 and R_2, which become isomorphic as modules over R_{12} after tensoring. In general, the P_i can not be patched to give an object of finite projective dimension over R. The necessary and sufficient condition for the P_i to be patchable is that they become isomorphic in the *derived category* over R_{12}, after applying the full (left derived) tensor product functor. (The formalism of derived categories is sketched in the body of the paper.)

The definition of the higher Chow groups involves modules. But to prove theorems about them—at least with the methods that I have used—it is necessary to reformulate everything in terms of complexes and derived categories. This suggests that it might be instructive to go back to basics and construct a theory in which complexes rather than modules are the fundamental objects of study.

At the same time, it seems useful to lift Bloch's construction from the level of abelian groups to the level of topological spaces, allowing the abelian groups to reemerge as appropriate homotopy groups. This allows the use of powerful techniques from topology that are not available in pure algebra.

I have recently introduced a construction that accomplishes both of these goals, taking complexes as primary and using topological methods. The result is a triply indexed family of invariants $KH_m^{j/j+1}(R,n)$ that generalize the higher Chow groups in view of the formula

$$KH_0^{j/j+1}(R,n) = Ch^j(R,n)$$

for all j and n.

At the same time, there naturally emerges a related "Karoubi-Villamayor" theory consisting of a doubly indexed family of invariants $KV_n^{j/j+1}(R)$. The Karoubi-Villamayor theory comes automatically equipped with an Atiyah- Hirzebruch spectral sequence converging from KV groups to algebraic K- theory. In addition, it is delicately intertwined with the KH theory, and there is in particular a non-trivial functorial map

$$KV_n^{j/j+1}(R) \to Ch^j(R, n).$$

Thus while the higher Chow groups themselves are not yet known to fit into an Atiyah-Hirzebruch sequence, they are at least related to something that does.

This paper is organized as follows: In Section 1, I describe the higher Chow groups. In Section 2, I describe the conjectural world of motivic cohomology. In Section 3, I describe the general strategy for establishing a relationship between higher Chow theory and algebraic K-theory (which, if fully successful, would show that the higher Chow groups *are* a motivic cohomology theory). In particular, I explain why it is necessary to have a good patching theory for objects of finite projective dimension. In Section 4 I explain why patching theory does *not* work well for modules, and why it *does* work well for complexes. In Section 5, motivated by the emergence of complexes as natural objects of study, I introduce the new families of invariants KH and KV, and discuss their relationships with K-theory, higher Chow groups and motivic cohomology. There is also a brief appendix, describing some research that is closely related to the ideas in the body of the paper.

This paper is entirely expository. Most sections stand on their own, referring to previous sections only for motivation, which the self-motivated reader might find unnecessary. Thus a reader interested only in patching, for example, could choose to read only Section 4, with an occasional backward glance to clarify notation.

It should be noted that the idea of defining K-theoretic invariants in terms of complexes, rather than modules, has a long history, beginning with Grothendieck. Waldhausen made categories of complexes the basis for his vast generalization of algebraic K-theory in [Wa]. The whole approach culminates in the great triumph of [TT], where it is indispensable to the formulation of a full-blown localization theorem for algebraic K-theory, which had been much sought after and highly elusive.

1. HIGHER CHOW GROUPS

In this section, I will describe the higher Chow groups of a commutative ring.

Higher Chow groups were invented by Spencer Bloch around 1979. The fundamental reference for their basic properties is [B].

1.1. The Simplicial Ring Associated to a Ring. Let R be any commutative ring and let $R^{(n)}$ be the ring $R[t_0, \ldots, t_n]/(\sum t_i = 1)$. (Obviously, $R^{(n)}$ is isomorphic to the polynomial ring $R[t_1, \ldots, t_n]$.)

Fixing n, we define a sequence of R-algebra maps

$$d_0, \ldots, d_n : R^{(n)} \to R^{(n-1)}$$

as follows:

$$d_i(t_j) = \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases}$$

We define another sequence of R-algebra maps

$$s_0, \ldots, s_n : R^{(n)} \to R^{(n+1)}$$

as follows:

$$s_i(t_j) = \begin{cases} t_j & \text{if } j < i \\ t_j + t_{j+1} & \text{if } j = i \\ t_{j+1} & \text{if } j > i \end{cases}$$

Notice that a given symbol d_i or s_i has been assigned infinitely many distinct meanings, one for each $n \geq i$. This is standard notation and rarely a source of confusion in practice.

The rings $R^{(n)}$, together with all of the maps d_i and s_i, form an example of a *simplicial ring*. In general, a simplicial ring (or simplicial group, or simplicial set, etc.) consists of a family of rings (or groups or sets, etc.) indexed by the non-negative integers, together with families of maps $\{d_i\}$ and $\{s_i\}$ subject to certain axioms. The most painless source of information about simplicial objects is [C].

1.2. Some Notation. Continuing to use the notation of 1.1, let M be an $R^{(n)}$-module. For any $i \in \{0, \ldots n\}$, view $R^{(n-1)}$ as an $R^{(n)}$-algebra via the homomorphism d_i. Then set

$$d_i(M) = M \otimes_{R^{(n)}} R^{(n-1)}$$

so that $d_i(M)$ is an $R^{(n-1)}$- module.

Similarly, view $R^{(n+1)}$ as an $R^{(n)}$-algebra via the homomorphism s_i and set

$$s_i(M) = M \otimes_{R^{(n)}} R^{(n+1)}.$$

1.3. Codimension. If M is a module over $R^{(n)}$ (or for that matter over any commutative ring), I will use the word *codimension* to mean the height of its annihilator. That is, if M is an $R^{(n)}$-module, we write

$$codim_{R^{(n)}}(M) = height_{R^{(n)}}(Annih(M)).$$

For each subset $I = \{i_1, \ldots, i_k\} \subset \{0, \ldots, n\}$, we write (t_I) for the ideal $(t_{i_1}, \ldots, t_{i_k}) \subset R^{(n)}$. If I is any such subset, and if M is any $R^{(n)}$-module, then exactly one of the following three conditions must hold:

(i) $codim_{(R^{(n)}/(t_I))}(M/t_IM) = j$

or (ii) $codim_{(R^{(n)}/(t_I))}(M/t_IM) < j$

or (iii) $M/t_IM = 0$.

We say that M is a *proper* $R^{(n)}$-module if for every $I \subset \{0, \ldots, n\}$, either (i) or (iii) holds. Roughly, M is proper if the act of modding out an ideal of the form (t_I) never causes the codimension of M to decrease.

In a sense that can be made precise using concepts from algebraic geometry, "most" $R^{(n)}$-modules are proper.

An easy consequence of the definitions is:

Proposition. *If M is proper of codimension j, then so is each $d_i(M)$.*

1.4. The Groups $Z^j(R, n)$. Now let R be a noetherian commutative ring. For any non-negative integers j and n, we define $Z^j(R, n)$ to be the free abelian group on the set of symbols

$$\{[R^{(n)}/P] \mid P \text{ is a prime ideal in } R^{(n)} \text{ and } R^{(n)}/P \text{ is a proper } R^{(n)}\text{-module of codimension } j\}.$$

Thus a typical element of $Z^j(R, n)$ is of the form

$$\sum_i n_i[R^{(n)}/P_i]$$

where the n_i are integers and the P_i are proper prime ideals of codimension j.

Fixing j, n, and $i \in \{0, \ldots, n\}$, we define a map

$$\begin{aligned} d_i : Z^j(R,n) &\longrightarrow Z^j(R,n-1) \\ [R^{(n)}/P] &\mapsto \sum_Q length_{R_Q^{(n-1)}}\big(d_i(R^{(n)}/P)_Q\big) \cdot [R^{(n-1)}/Q] \end{aligned}$$

The sum here is over all prime ideals Q in $R^{(n-1)}$ that are proper and of codimension j. It is easily verified that all but a finite number of the coefficients are zero, so that the sum makes sense. The $R^{(n-1)}$-module $d_i(R^{(n)}/P)$ is as defined in 1.2.

The coefficient $length_{R_Q^{(n-1)}}\big(d_i(R^{(n)}/P)_Q\big)$ can be interpreted as the intersection multiplicity of $R^{(n)}/P$ with $R^{(n)}/(t_i)$ at the prime ideal $d_i^{-1}(Q)$.

We also define maps

$$\begin{aligned} s_i : Z^j(R,n) &\longrightarrow Z^{j+1}(R,n) \\ [R^{(n)}/P] &\mapsto [s_i(R^{(n)}/P)] = [R^{(n+1)}/s_i(P)R^{(n+1)}] \end{aligned}$$

which makes sense because $R^{(n+1)}/s_i(P)R^{(n+1)}$ is a proper $R^{(n+1)}$-module of codimension j.

For fixed j, the groups $Z^j(X,n)$ together with the maps d_i and s_i just defined form a simplicial abelian group that we will denote by $Z^j(X,\bullet)$.

1.5. The Higher Chow Groups. To a simplicial object, there is an associated sequence of *homotopy groups*, which are fully described in [C]. Fortunately, the homotopy groups of a simplicial abelian group have a particularly simple description.

Fix j and consider the simplicial abelian group $Z^j(X,\bullet)$. We construct an associated complex

$$Z^j(R,0) \xleftarrow{d} Z^j(R,1) \xleftarrow{d} Z^j(R,2) \xleftarrow{d} \cdots \tag{1.5.1}$$

where the arrow $d : Z^j(R,n) \to Z^j(R,n-1)$ is defined by the formula

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

We will use the same notation $Z^j(R,\bullet)$ to denote the complex (1.5.1) and the simplicial abelian group from which it is derived. Then the n^{th} homotopy group $\pi_n(Z^j(R,\bullet))$ turns out to be naturally isomorphic to the n^{th} homology group of this associated complex.

We define the *higher Chow groups* $Ch^j(R, n)$ by the formula

$$\begin{aligned} Ch^j(R, n) &= \pi_n(Z^j(R, \bullet)) \\ &\approx H_n(Z^j(R, \bullet)). \end{aligned} \tag{1.5.2}$$

Although this definition makes sense for any noetherian ring R, *we will henceforth assume that all rings under consideration are regular rings.*

1.6. Some Examples. For the reader who has understood the definitions, it should be easy to check that (under our continuing assumption that R is regular)

$$Ch^1(R, 0) \approx Pic(R).$$

More generally, for the reader familiar with the classical Chow groups $Ch^j(R)$ (see [F] for an extensive discussion of these) it should be easy to check that

$$Ch^j(R, 0) \approx Ch^j(R)$$

for all j.

A slightly more difficult computation shows that

$$Ch^1(R, 1) \approx R^*$$

where R^* is the group of units in R.

When $R = k$ is a field, work of Suslin relates the higher Chow groups to the Milnor K-groups $K^M(k)$:

$$Ch^m(k, m) \approx K^M_m(k)$$

for all m. (For information on the Milnor K- groups and their relations with quadratic forms, see [Mi2] or [BT].)

1.7. Analogy With Singular Homology. Let me be explicit about why the higher Chow groups can be viewed as the algebraist's analogue of the topologist's singular homology (or cohomology) groups. (For smooth varieties, or for regular rings, homology and cohomology should be isomorphic via Poincaré duality. A general formalism for theories satisfying Poincaré duality is laid out in [BO].)

First recall the construction of singular homology. We start with the "standard simplices" $S^n = \{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} | \sum t_i = 1\}$. Given a topological space X we let $C(X, n)$ be the free abelian group on the set of continuous functions $S^n \to X$. We define maps $d_0, \ldots, d_n : C(X, n) \to C(X, n-1)$ and $s_0, \ldots, s_n : C(X, n) \to C(X, n+1)$ by formulas like those of Section 1.1. These make $C(X, \bullet)$ into a simplicial abelian group, and we define singular homology by setting

$$H_n(X) = \pi_n(C(X, \bullet)).$$

As in Section 1.5, we can construct an associated complex, also denoted $C(X, \bullet)$, in which the differential is $d = \sum(-1)^i d_i$. There is then a natural isomorphism

$$H_n(X) \approx H_n(C(X, \bullet)).$$

Let us imitate this construction in the algebraic setting. In the category of finitely generated algebras over a fixed field k, the analogue of S^n is the ring $k[t_0, \ldots, t_n]/(\sum t_i - 1)$. In the category of commutative rings, the analogue of S^n is the ring $\mathbf{Z}[t_0, \ldots, t_n]/(\sum t_i - 1)$. I will write $\Sigma^n = k[t_0, \ldots, t_n]/(\sum t_i - 1)$, where k might be a field or $\mathbf{Z}$, depending on context.

Now let R be a k-algebra. To construct "singular homology" groups for R, we want to consider something like the set of k-scheme maps

$$f^* : Spec(\Sigma^n) \to Spec(R),$$

or equivalently k-algebra maps

$$f : R \to \Sigma^n.$$

Such a map can be extended uniquely to a k-algebra map

$$\hat{f} : R[t_0, \ldots, t_n]/(\textstyle\sum t_i - 1) \to \Sigma^n \tag{1.7.1}$$

and f is uniquely determined by $ker(\hat{f})$, which is a prime ideal $P \subset R[t_0, \ldots, t_n]/(\sum t_i - 1) = R^{(n)}$.

Thus every map

$$f^* : Spec(\Sigma^n) \to Spec(R)$$

gives rise to a prime ideal $P \subset R^{(n)}$, or equivalently to a module of the form $R^{(n)}/P$. Moreover, the module $R^{(n)}/P$ is always proper (in the sense of Section 1.3) and of codimension $d = dim(R)$. It follows that a good analogue of the topological construction $C(X, n)$ is the free abelian group on generators of the form $[R^{(n)}/P]$ where $R^{(n)}/P$ is proper and of codimension d, i.e. $Z^d(R, n)$ as defined in Section 1.4. The remainder of the construction, already carried out in Section 1.5, now proceeds exactly as in the topological case, the result being the group $Ch^d(R, n)$ as an analogue of the topologist's $H_n(X)$.

The same argument can be put in geometric form: A map $f^* : Spec(\Sigma^n) \to Spec(R)$ can be identified with its graph $\Gamma_f \subset Spec(\Sigma^n) \times Spec(R) \approx Spec(R^{(n)})$. Γ_f is a closed irreducible subset of $Spec(R^{(n)})$ and so can be identified with $Spec(R^{(n)}/P)$ for some prime ideal P; this constructs the same class $[R^{(n)}/P]$ as does the algebraic argument of the preceding paragraph.

Note, however, that not every closed irreducible codimension-d subset in $Spec(\Sigma^n) \times Spec(R)$ is the graph of a function; equivalently, not every prime ideal P of height d in $R^{(n)}$ arises from a map (1.7.1). The groups $Z^d(R, n)$ contain much more than just the graphs of functions. In fact, the typical element of $Z^d(R, n)$ can be thought of as the graph of a *correspondence*, or multi-valued function. In algebra, unlike in topology, functions alone do not suffice for the construction of interesting homological invariants.

Another way in which the algebraic situation differs from the topological one is in the appearance of the groups $Z^j(R, n)$ for $j \neq d$. To understand the role of these groups, let X be a smooth complex variety of complex dimension d (hence of real dimension $2d$). Then for n even, subvarieties of complex dimension $n/2$ (hence real dimension n) are represented by cycle classes in $H_n(X)$. When R is a ring of dimension d, subschemes of $Spec(R)$ having dimension $n/2$ are represented by classes in $Z^{d-(n/2)}(R, 0)$. This suggests that $Ch^{d-(n/2)}(R, 0)$ should also be something like an analogue of $H_n(X)$.

We have argued that $Ch^d(R, n)$ is "like" $H_n(X)$ because its elements are represented by (possibly multi-valued) functions from an "n-simplex" to $Spec(R)$; and that $Ch^{d-(n/2)}(X, 0)$ is "like" $H_n(X)$ because its elements are represented by $(n/2)$-dimensional subschemes of $Spec(R)$. Interpolating linearly between these two intuitions, we expect that as j varies, the groups $Ch^j(X, 2j - 2d + n)$ should all bear some resemblance to an n^{th} homology group for R. Since Poincaré duality should provide an isomorphism $H_n(R) \approx H^{2d-n}(R)$, we can rephrase this as follows:

The groups $Ch^j(X, 2j-n)$ are like pieces of a cohomology group $H^n(R)$. (1.7.2)

2. MOTIVIC COHOMOLOGY

In modern algebraic geometry and commutative algebra, there are many cohomology theories. Singular cohomology, étale cohomology, crystalline cohomology, and Deligne cohomology are a few of many examples. Grothendieck envisioned the possibility that all of these cohomology theories are manifestations of a single *universal* cohomology theory, which he called *motivic cohomology*.

2.1. The Vision. Let $\mathcal{C}$ be a well-behaved category of objects having cohomology; for example, the category of smooth algebraic varieties over some base field k. Then the motivic cohomology theory $H_{\mathcal{M}}$ for $\mathcal{C}$ should consist of something like the following. First, we need a category $\mathcal{H}$ in which $H_{\mathcal{M}}$ can take its values. $\mathcal{H}$ should be an abelian category, and its objects should be graded. The motivic cohomology theory itself should then be a functor

$$H_{\mathcal{M}} : \mathcal{C} \to \mathcal{H}$$

having all of the desirable properties of a cohomology theory (like Künneth formulas and Poincaré duality). By applying $H_{\mathcal{M}}$ to an object $C \in \mathcal{C}$ and then projecting onto the graded pieces, we get the *motivic cohomology groups* $H^i_{\mathcal{M}}(C)$.

(At this level of generality the phrase "cohomology group" is perhaps a tad inappropriate; these "groups" are not necessarily groups at all, but objects in the abstract abelian category $\mathcal{H}$.

The key property we seek is the following: $H_{\mathcal{M}}$ should be defined in such a way that if $H : \mathcal{C} \to \mathcal{A}$ is any other cohomology theory, taking its values in a graded abelian category $\mathcal{A}$, then H factors uniquely through motivic cohomology; that is, there exists a unique functor $F_H : \mathcal{H} \to \mathcal{A}$ such that H is equal to the composition

$$\mathcal{C} \xrightarrow{H_{\mathcal{M}}} \mathcal{H} \xrightarrow{F_H} \mathcal{A}.$$

Moreover, the map F_H is required to be an *additive* functor between abelian categories and hence relatively easy to understand. All of the deepest mathematics is to be encoded in the single family of functors $H^i_{\mathcal{M}}$.

Things would be even nicer if $\mathcal{H}$ could be taken to be a *semisimple* category, i.e. one in which every object is a direct sum of simple objects. It would follow that every additive functor from $\mathcal{H}$ is determined entirely by its action on the subcategory of simple objects. This would make the functors F_H, and therefore the cohomology theories H, appear even more elementary.

For an elementary introduction to this point of view, see [Ma], where Manin– -following ideas of Grothendieck—constructs an approximation to the category $\mathcal{H}$ and the functor of motivic cohomology. Essentially he enlarges the category of varieties by throwing in "images" for all of the idempotent maps. The motivic cohomology of a variety is then the variety itself, thought of as an object in the enlarged category. Unfortunately, the conjecture that Manin's $\mathcal{H}$ is abelian appears extraordinarily difficult. Fortunately, $\mathcal{H}$ is still sufficient for some remarkable applications. Using it, Manin gives an easy and beautiful proof of the Weil conjectures for a nonsingular cubic hypersurface in $\mathbf{P}^4$.

2.2. The Atiyah-Hirzebruch Spectral Sequence. To see what sorts of properties we can hope for from motivic cohomology of algebraic varieties or of commutative rings, we attempt to draw analogies from what we know about singular cohomology of topological manifolds.

Let X be a manifold with singular cohomology groups $H^n(X)$. Then X has topological K-groups $K^n(X)$, and there is an *Atiyah-Hirzebruch spectral sequence*

$$E_2^{p,q} = H^{p-q}(X) \Longrightarrow K^{p+q}(X). \tag{2.2.1}$$

The differential d_r maps $E_r^{p,q}$ to $E_r^{p+r,q-r-1}$. (All spectral sequences in this paper will be indexed in this way.)

In the E_2 complex, what groups appear along the diagonal $p+q=n$? The answer is: all of the even-dimensional cohomology groups if n is even, and all of the odd-dimensional cohomology groups if n is odd. This reflects the fact that the topological K-group $K^n(X)$ depends only on the parity of n.

However, in algebra the situation is more complicated, because algebraic K- theory, unlike topological K-theory, is not periodic. A spectral sequence with an E_2 complex that is "periodic" like that of (2.2.1) can not converge to algebraic K-theory. (See [BMS] for elaboration of this point.)

This suggests that if there is to be an algebraic analogue of the Atiyah- Hirzebruch spectral sequence, the E_2 terms must actually depend on both of the indices p and q, and not just on the difference $p - q$. If there exists an "Atiyah-Hirzebruch" spectral sequence with motivic cohomology groups appearing at the E_2 level, then motivic cohomology must actually be a *doubly* indexed theory; that is, the functor $H^n_{\mathcal{M}}$ should split up as a direct sum of functors $H^n_{\mathcal{M}}(-, j)$.

Thus a (perhaps overly optimistic) conjecture is that for some class of algebraic varieties X there exists a well-behaved doubly indexed cohomology theory $H^n_{\mathcal{M}}(-, j)$, and a spectral sequence

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, -q) \Longrightarrow K_{-p-q}(X). \tag{2.2.2}$$

2.3. Motivic Complexes. Lichtenbaum and Beilinson have proposed a (conjectural) framework for constructing motivic cohomology. For a commutative ring R (at least when R is regular and of finite type over a field) they conjecture the existence of *motivic complexes* $\mathbf{Z}(r)^\bullet$ $(r \geq 0)$. These motivic complexes are complexes of abelian groups, indexed cohomologically (that is, the differentials increase degree), defined up to quasi-isomorphism, that are functorial in R. (The notation suppresses the fact that $\mathbf{Z}(j)^\bullet$ depends on R.)

The motivic complexes are required to satisfy a family of axioms that make it reasonable to define the motivic cohomology of R by the formula

$$H^n_{\mathcal{M}}(R, j) = H^n(\mathbf{Z}(j)^\bullet). \tag{2.3.1}$$

In particular, one of the axioms is the existence of an Atiyah-Hirzebruch-type spectral sequence

$$E_2^{p,q} = H^{p-q}(\mathbf{Z}(-q)^\bullet) \Longrightarrow K_{-p-q}(R). \tag{2.3.2}$$

(Compare this with (2.2.2).)

The Lichtenbaum/Beilinson conjectures appear to be very difficult. The complex $\mathbf{Z}(0)^\bullet$ must consist of just $\mathbf{Z}$ in degree zero and the complex $\mathbf{Z}(1)^\bullet$ is a complex whose only cohomology is R^* in degree 1 and $Pic(R)$ in degree 2. By contrast, Lichtenbaum's construction of a candidate for $\mathbf{Z}(2)^\bullet$ in [Li] is a major piece of work.*

2.4. Motivic Cohomology and Ext Groups. Suppose that there exist motivic complexes as in Section 2.3, and that we use them to define motivic cohomology via formula 2.3.1. Then (making use of the fact that $\mathbf{Z}(0)^\bullet = \mathbf{Z}$) we can rewrite

$$H^n_{\mathcal{M}}(R, j) = Ext^n_{\mathcal{H}}(\mathbf{Z}(0)^\bullet, \mathbf{Z}(j)^\bullet) \tag{2.4.1}$$

where the Ext is computed in some abelian category $\mathcal{H}$ where the motivic complexes live. (Yes, this is as vague as it appears.) Ideally, $\mathcal{H}$ should be a category of objects endowed with all of the structures that naturally appear on cohomology groups, such as mixed Hodge structures. When these structures arise in any cohomology theory, they will then be specializations of the structures that occur naturally on the motivic groups.

3. HIGHER CHOW GROUPS AND ALGEBRAIC K-THEORY.

3.1. Some History. Around 1979, Spencer Bloch invented the higher Chow groups and outlined a program for relating them to algebraic K-theory. For a regular ring R, he conjectured the existence of a filtration on $K_n(R)$ with associated graded groups closely related to the groups $Z^j(X, n)$ that are defined in Section 1.5.

In the early 1980's, I attempted to carry out Bloch's program and was led to a more precise formulation of the relationship that should hold between K- theory and higher Chow groups. Specifically, I conjectured that for a regular ring R of finite type over a

* A more natural—and more usual—formulation is to let $\mathcal{Z}(j)^\bullet$ be the sheafification of the complex $\mathbf{Z}(j)$ for the Zariski topology and set $H^n_{\mathcal{M}}(R, j) = \mathbf{H}^n(Spec(R), \mathcal{Z}(j)^\bullet)$. (This is instead of 2.3.1.) The two formulations are equivalent if one takes my $\mathbf{Z}(j)$ to be (up to quasi- isomorphism) the complex of global sections of an injective resolution of the sheafification of the usual $\mathbf{Z}(j)$. My only purpose in departing from the norm is to minimize talk of sheaves and hypercohomology.

field, there is a spectral sequence

$$E_2^{p,q} = Ch^{-q}(R, -p-q) \Longrightarrow K_{-p-q}(R). \tag{3.1.1}$$

(To be entirely accurate, I did not "conjecture"; I announced a theorem. After some time, it became apparent that the word *conjecture* described the situation more accurately.)

In 1982, I mailed a preprint containing the then-theorem/now-conjecture (3.1.1) to C. Soulé in Paris. At almost exactly the same time, he received a letter from Beilinson conjecturing the existence of motivic complexes $\mathbf{Z}(j)^\bullet$ and a spectral sequence (2.3.2). (Beilinson's ideas appear in [Be].)

Comparing (3.1.1) to (2.3.2), and in light of the definition (1.5.2) Soulé was led to the obvious conjecture that after an appropriate reindexing

$$\mathbf{Z}(j)^\bullet = Z^j(R, \bullet).$$

After mucking around to get the indices right, one refines the conjecture to

$$\mathbf{Z}(j)^n = Z^j(R, 2j-n).$$

(Note in particular that this converts the homologically indexed complex $Z^j(R, \bullet)$ to a cohomologically indexed complex via the introduction of a minus sign.)

Notice that a consequence of this conjecture is

$$H^n_{\mathcal{M}}(R, j) \approx Ch^j(R, 2j-n) \tag{3.1.2}$$

as suggested by (1.7.2).

The conjecture that $\mathbf{Z}(j)^\bullet = Z^j(R, 2j-\bullet)$ now has many adherents, but appears to be extremely difficult. In [RCAKT] I constructed a filtration on $K_n(R)$ such that the associated graded groups are subquotients of the groups $Ch^\bullet(R, \bullet)$ in the way that would be expected if the spectral sequence (3.1.1) exists. In [B], Bloch developed many desirable properties of the higher Chow groups and showed in particular that up to torsion, they are actually isomorphic to the associated graded of a filtration on higher K-theory. (In fact, the filtration is the γ-filtration, which has long been of interest to K-theorists.) This is in accordance with the expectation that the Atiyah-Hirzebruch spectral sequence should degenerate up to torsion, as it does in topology.

3.2. Relating K-Theory to Higher Chow Groups. The problem has been to relate the higher Chow groups of Section 1.5 to higher algebraic K-theory. Why should one expect any relation at all between these two sets of invariants? Let me begin by sketching the original program as Bloch outlined it ten years ago.

Let us take an element $\xi \in Ch^j(R,n)$ and think about how it could be mapped to an element of $K_n(R)$. From the definition (1.5.2), ξ is represented by an element

$$z \in ker\left(d = \sum_i (-1)^i d_i : Z^j(R,n) \to Z^j(R,n-1)\right).$$

In fact, one checks easily that z can be chosen so that $d_i(z) = 0$ for all i, and we shall assume that it has been chosen in that way.

We can write z as $z = z_+ - z_-$, where

$$\begin{aligned} z_+ &= \sum_j n_j [R^{(n)}/P_j] \\ z_- &= \sum_k m_k [R^{(n)}/P_k] \end{aligned}$$

and all of the n_j and m_k are non-negative integers. From these we construct two $R^{(n)}$-modules:

$$\begin{aligned} M_+ &= \bigoplus_j (R^{(n)}/P_j)^{n_j} \\ M_- &= \bigoplus_k (R^{(n)}/P_k)^{m_k}. \end{aligned} \tag{3.2.1}$$

Next we construct a pullback diagram of commutative rings, in which both of the maps $R^{(n)} \to R^{(n)}/(\Pi t_i)$ are equal to the canonical surjection:

$$\begin{array}{ccc} S^{(n)} & \longrightarrow & R^{(n)} \\ \downarrow & & \downarrow \\ R^{(n)} & \longrightarrow & R^{(n)}/(\Pi t_i) \end{array}. \tag{3.2.2}$$

We view M_+ as a module over the copy of $R^{(n)}$ in the upper right-hand corner and M_- as a module over the copy of $R^{(n)}$ in the lower left-hand corner. The condition that $d_i(z) = 0$ for all i comes very close to saying that $M_+/\big((\Pi t_i)M_+\big) \approx M_-/\big((\Pi t_i)M_-\big)$

as modules over $R^{(n)}/(\Pi t_i)$. (What the condition actually says is that the two (cyclic) quotient modules have the same length after localizing at any prime ideal.)

Suppose that after a small amount of fiddling around, M_+ and M_- can be "adjusted" so that the two quotient modules are isomorphic. (If you're looking for precision, see, e.g. [RCAKT].) In that case, one can choose an isomorphism and construct a pullback module M over $S^{(n)}$. We think of M as being given by "patching" M_+ to M_-.

Suppose also that by a stroke of fortune M has finite projective dimension over $S^{(n)}$. In that case, there is an associated class $[M] \in K_0(S^{(n)})$. (Choose a finite projective resolution $P_\bullet \to M$ and set $[M] = \sum(-1)^i[P_i]$; $[M]$ is well-defined by Schanuel's Lemma.)

A calculation using the Karoubi-Villamayor approach to $K-$theory reveals that

$$K_0(S^{(n)}) \approx K_n(R) \oplus K_0(R).$$

Projection onto the first factor gives the class in $K_n(R)$ that we seek.

3.3. Some Technical Difficulties. The approach of Section 3.2 is essentially Bloch's original program. The technical barriers to carrying it out are formidable, and require that the program be modified in several directions. Let me focus here on the problem of forcing M to have finite projective dimension.

Suppose that M_+ and M_- are given by the formulas (3.2.1), that they become isomorphic mod (Πt_i), and that they are patched to form an $S^{(n)}$-module M. Then it is not hard to show that the projective dimension of M is indeed finite. The key observations needed to derive this result are: First, M_+ and M_- have finite projective dimension over $R^{(n)}$ (because $R^{(n)}$ is regular); and second,

$$Tor_1^{R^{(n)}}\left(M_+, R^{(n)}/(\Pi t_i)\right) = Tor_1^{R^{(n)}}\left(M_-, R^{(n)}/(\Pi t_i)\right) = 0. \tag{3.3.1}$$

(This follows from the fact that all of the modules $R^{(n)}/P_j$ and $R^{(n)}/P_k$ are proper in the sense of Section 1.3.) By an easy argument (found, e.g. in [PMFPD]), these observations suffice to show that M has finite projective dimension.

Unfortunately, as we have noted, the original modules $M_+/(\Pi t_i M_+)$ and $M_-/(\Pi t_i M_-)$ of (3.2.1) can fail to be isomorphic, in which case M_+ and M_- must be massaged a bit before they can be patched. Typically, the massaging process destroys (3.3.1) and with it the automatic finiteness of projective dimension for M.

In fact, there is also a far more serious (though subtler) problem along these lines. Because of the "massaging" process, the module M ends up not being well-defined. It is well-defined only up to the K-theory class of an "error module" N. In order for the theory to work out as it should, N must also be constructed by the same sort of patching process that yields M. But there is no reason to expect that N is of this form.

What is needed, then, is a good patching theory for objects of finite projective dimension. In terms of diagram (3.2.2), there are two requisites for such a theory. First, appropriate patching data should yield an object of finite projective dimension over $S^{(n)}$. Second, every object of finite projective dimension over $S^{(n)}$ should arise via patching.

In Section 4.1 I will elaborate on why modules are the wrong candidates for the "objects" of the preceding paragraph. That elaboration will motivate the introduction of the right candidates in Section 4.2.

4. PATCHING

4.1. Obstructions to Finite Projective Dimension. Let me try to give a little more insight into why the "patched" module M of Section 3.2 can fail to have finite projective dimension. To summarize the setup: We have a pullback diagram (3.2.2), and modules M_+ and M_- over the two copies of $R^{(n)}$; we assume that M_+ and M_- become isomorphic over $R^{(n)}/(\Pi t_i)$, and we construct the $S^{(n)}$ module M by patching. What is the obstruction to finite projective dimension for M?*

I will write $R_+^{(n)}$ and $R_-^{(n)}$ for the two copies of $R^{(n)}$ with their induced structure as $S^{(n)}$-modules. The first observation is that $R_+^{(n)}$ and $R_-^{(n)}$ are quotients of $S^{(n)}$ by principal ideals (x_+) and (x_-) that are equal to each others' annihilators. Consequently there are

* For the discussion of patching problems, (3.2.2) could be replaced with a far more general pullback diagram; in particular, there is no need for the rings in the northeast and southwest corners to be the same. However, I will restrict the exposition to the case at hand.

exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{x_+} & S^{(n)} & \xrightarrow{x_-} & S^{(n)} & \xrightarrow{x_+} & S^{(n)} & \to & R_+^{(n)} & \to & 0 \\
\cdots & \xrightarrow{x_-} & S^{(n)} & \xrightarrow{x_+} & S^{(n)} & \xrightarrow{x_-} & S^{(n)} & \to & R_-^{(n)} & \to & 0.
\end{array}$$

Computing with these sequences, we discover that Tor is periodic, i.e.

$$\cdots \;=\; Tor_5^{S^{(n)}}(M, R_+^{(n)}) \;=\; Tor_3^{S^{(n)}}(M, R_+^{(n)}) \;=\; Tor_1^{S^{(n)}}(M, R_+^{(n)})$$

and

$$\cdots \;=\; Tor_6^{S^{(n)}}(M, R_+^{(n)}) \;=\; Tor_4^{S^{(n)}}(M, R_+^{(n)}) \;=\; Tor_2^{S^{(n)}}(M, R_+^{(n)})$$

and likewise with $R_+^{(n)}$ replaced by $R_-^{(n)}$. In particular, if M has finite projective dimension, then sufficiently high Tor's must vanish, and so it follows that

$$Tor_*^{S^{(n)}}(M, R_+^{(n)}) = Tor_*^{S^{(n)}}(M, R_-^{(n)}) = 0 \qquad \text{for all } * > 0. \tag{4.1.1}$$

Now let $P_\bullet \to M$ be a finite projective resolution. Tensor with $R_+^{(n)}$ and $R_-^{(n)}$ to get complexes

$$\begin{aligned} P_{\bullet +} &\longrightarrow M_+ \\ P_{\bullet -} &\longrightarrow M_- \end{aligned} \tag{4.1.2}$$

which are projective resolutions over $R_+^{(n)}$ and $R_-^{(n)}$ by (4.1.1), and which satisfy

$$P_{\bullet +} \otimes_{R_+^{(n)}} \left(R^{(n)}/(\Pi t_i)\right) \approx P_{\bullet -} \otimes_{R_-^{(n)}} \left(R^{(n)}/(\Pi t_i)\right).$$

Using these resolutions to compute Tor, we are able to conclude that for any $R^{(n)}/(\Pi t_i)$-module N,

$$Tor_*^{R^{(n)}}(M_+, N) \quad \approx \quad Tor_*^{R^{(n)}}(M_-, N) \qquad \text{for all } * > 0. \tag{4.1.3}$$

Indeed, each side of the equation is isomorphic to $Tor_*^{S^{(n)}}(M, N)$.

Condition (4.1.3) is a necessary condition, in terms of M_+ and M_-, for the patched module M to have finite projective dimension. But it is also a condition that often fails (for arbitrary M_+ and M_-). Thus module patching is not a promising route to constructing K-theory classes.

4.2. From Modules to Complexes. If modules do not patch well, then what does? Condition (4.1.2) provides the clue. It says that in order for M_+ and M_- to patch well, they must admit finite projective resolutions that become isomorphic as complexes when they are reduced mod (Πt_i). This suggests that it is the complexes, rather than the modules, that are really being patched.

Moreover, we know by the classic result of Milnor in [Mi] that *projective* modules patch nicely along isomorphisms, and it follows that *complexes* of projective modules patch nicely along isomorphisms. In other words, patching works well for complexes of projectives. By the preceding paragraph, patching works well *only* when it can be reinterpreted in terms of complexes of projectives. It pretty much follows that complexes of projectives are the only reasonable objects of study in this context.

But it is still convenient to be able to work with modules and with complexes of non-projective modules. It turns out that the right compromise is to formulate a patching theorem for *perfect complexes*.

We work always over a noetherian ring. A complex $A_\bullet$ of modules is *perfect* if there exists a bounded complex of finitely generated projective modules $P_\bullet$ and a quasi-isomorphism $P_\bullet \to A_\bullet$. (A map of complexes is a *quasi-isomorphism* if it induces isomorphisms on all homology modules.) If M is a finitely generated module of finite projective dimension, then we identify M with the perfect complex consisting of M in degree zero and zeros elsewhere. (To see that this is perfect, let $P_\bullet$ be a finite projective resolution of M.)

It is convenient to formulate things in terms of the *derived category*, which for our purposes means the following: Start with the category of perfect complexes. Identify maps that are chain homotopy equivalent. Now formally invert all quasi-isomorphisms (via a process analogous to that of localizing a ring at a multiplicative set). The easiest place to read the details of this construction is [H].

If R is a noetherian ring, write $\mathcal{M}(R)$ for the category of finitely generated R-modules and $\mathcal{D}(R)$ for the derived category described in the preceding paragraph. If $R \to S$ is a homomorphism, then the functor

$$- \otimes_R S : \mathcal{M}(R) \to \mathcal{M}(S)$$

gives rise to a *left derived functor*

$$- \overset{L}{\otimes}_R S : \mathcal{D}(R) \to \mathcal{D}(S).$$

The left derived functor can be partially described as follows: Let $A_\bullet$ be a perfect complex over R. Choose a complex of finitely generated projective R-modules $P_\bullet$ such that there exists a quasi-isomorphism $P_\bullet \to A_\bullet$. Then up to quasi-isomorphism we have

$$A_\bullet \overset{L}{\otimes}_R S = P_\bullet \otimes {}_R S.$$

For the existence of a left derived functor with this property, the reader is referred once again to [H].

Refer once more to diagram (3.2.2) and suppose that we are given modules M_+ and M_- in the northeast and southwest corners, becoming isomorphic mod (Πt_i). This last condition can be restated as

$$M_+ \otimes_{R^{(n)}} \left(R^{(n)}/(\Pi t_i)\right) \quad \approx \quad M_- \otimes_{R^{(n)}} \left(R^{(n)}/(\Pi t_i)\right). \tag{4.2.1}$$

However, condition (4.2.1) says precisely that if M_+ and M_- can be patched to give a module of finite projective dimension over $S^{(n)}$, then they must satisfy the stronger condition

$$M_+ \overset{L}{\otimes}_{R^{(n)}} \left(R^{(n)}/(\Pi t_i)\right) \quad \approx \quad M_- \overset{L}{\otimes}_{R^{(n)}} \left(R^{(n)}/(\Pi t_i)\right). \tag{4.2.2}$$

Thus the right context for patching must be the derived category. In the derived category, every quasi-isomorphism of complexes becomes an isomorphism, so we must be permitted to patch along quasi-isomorphisms. We therefore define *patching data* to consist of perfect complexes $P_{\bullet+}$ and $P_{\bullet-}$ over the northeast and southwest corners of (3.2.2) and a quasi-isomorphism

$$\alpha : \overline{P}_{\bullet+} \to \overline{P}_{\bullet-}$$

where the overbar denotes reduction mod (Πt_i). The desired patching theorem is then:

Theorem 4.2.3. *There is a functor from the category of patching data to the category of perfect complexes over $S^{(n)}$. When $P_{\bullet+}$ and $P_{\bullet-}$ are single projective modules concentrated in degree zero, the image of $(P_{\bullet+}, P_{\bullet-}, \alpha)$ is the usual pullback module. Up to quasi-isomorphism, every perfect complex over $S^{(n)}$ arises in this way.*

Because of the identification of modules with perfect complexes concentrated in degree zero, the theorem contains a result on module patching. In particular, M_+ and M_- can be patched whenever (4.2.2) holds.

Theorem 4.2.3 is proved in [KTP], where I used it to construct natural maps

$$K_n(R^{(n)}/(\Pi t_i)) \to K_{n-1}(S^{(n)})$$

and (more importantly) similar maps with the rings replaced by categories of modules satisfying a prescribed upper bound on codimension. It should be noted that patching complexes along actual isomorphisms is a trivial exercise; patching along arbitrary *quasi*-isomorphisms requires a bit of work.

5. SOME NEW INVARIANTS

The higher Chow groups are essentially defined in terms of modules, and are related to higher K-theory via patching. But we have just seen that the natural context for patching is not the category of modules, but rather the category of perfect complexes. This suggests mimicking Bloch's higher Chow construction with the modules replace by complexes. At the same time, I want to lift the entire construction from the level of abelian groups to the level of topological spaces (with Bloch's abelian groups occurring as homotopy groups). I have carried out such a construction in [SF] and will report briefly on it here.

5.1. Higher Higher Chow Groups. Using the notation and definitions of Sections 1.1-1.3, let $\mathcal{M}^j(R,n)$ be the category of all those bounded complexes of finitely generated $R^{(n)}$-modules whose homology is annihilated by some ideal I with R/I proper and of codimension j. Using a construction of Waldhausen [W], we can associate to $\mathcal{M}^j(R,n)$ a topological space $K^j(R,n)$ whose homotopy groups are defined to be the algebraic K-groups of $\mathcal{M}^j(R,n)$:

$$K_m(\mathcal{M}^j(R,n)) = \pi_m(K^j(R,n)).$$

The maps d_i and s_i of Section 1.1 induce well-behaved functors among the categories $\mathcal{M}^j(R,n)$ and consequently continuous maps among the topological spaces $K^j(R,n)$. We will denote these induced maps by d_i and s_i also. Then for fixed j, the spaces $K^j(R,n)$ together with the maps d_i and s_i form a simplicial topological space.

For each j and n, inclusion of categories induces a map of simplicial spaces

$$K^{j+1}(R,n) \to K^j(R,n).$$

Using standard topological constructions we can construct a space $K^{j/j+1}(R,n)$ and a map $K^j(R,n) \to K^{j/j+1}(R,n)$ such that the sequence

$$K^{j+1}(R,n) \to K^j(R,n) \to K^{j/j+1}(R,n)$$

yields a long exact sequence of homotopy groups. For fixed j, the groups $K^{j/j+1}(R,n)$ also fit together naturally to form a simplicial space.

From the simplicial space $K^{j/j+1}(R,\bullet)$, we can construct for each non-negative integer m a simplicial abelian group by replacing each $K^{j/j+1}(R,\bullet)$ with its m^{th} homotopy group. This yields a sequence of simplicial abelian groups

$$K_m^{j/j+1}(R,\bullet).$$

The following theorem (proved in [SF]) justifies thinking of these simplicial abelian groups as "higher higher Chow constructions":

Theorem 5.1.1. $K_0^{j/j+1}(R,\bullet) \approx Z^j(R,\bullet)$.

The isomorphism is an isomorphism of simplicial abelian groups.

If we define a triply indexed family of "higher higher Chow groups" by

$$KH_m^{j/j+1}(R,n) = \pi_n(K_m^{j/j+1}(R,\bullet)),$$

then Theorem 5.1.1 implies that

$$KH_0^{j/j+1}(R,n) \approx Ch^j(R,n). \tag{5.1.2}$$

Thus the higher higher Chow groups generalize the (now classical) higher Chow groups of Section 1.5.

5.2. A Karoubi-Villamayor Theory. In section 5.1, we converted the simplicial space $K^{j/j+1}(R,\bullet)$ into a simplicial abelian group by applying the functor π_m to each of the consitutent spaces. We then defined higher higher Chow theory by computing the homotopy groups of this simplicial group.

But $K^{j/j+1}(R,\bullet)$, as a simplicial space, has its own homotopy groups. The construction of these groups is reminiscent of the Karoubi- Villamayor approach to K-theory, and

I think of them as a Karoubi- Villamayor theory for modules of codimension $\geq j$. I will write

$$KV_m^{j/j+1}(R) = \pi_m(K^{j/j+1}(R, \bullet).$$

There are many complicated relationships among the KV groups, the KH groups, and algebraic K-theory. Typically, these relations manifest themselves in spectral sequences; for example we have

$$E_2^{p,q} = KV_{-p-q}^{-q/-q+1}(R) \Longrightarrow K_{-p-q}(R) \tag{5.2.1}$$

$$E_2^{p,q} = KH_{-q}^{j/j+1}(R, -p) \Longrightarrow KV_{-p-q}^{j/j+1}(R) \qquad \text{for each fixed } j \tag{5.2.2}$$

$$E_1^{p,q} = KH_{r-p}^{p/p+1}(R, -q) \Longrightarrow H^{p+q}(Spec(R), \mathcal{K}_r) \qquad \text{for each fixed } r \tag{5.2.3}$$

(The abutment term in (5.2.3) is Zariski cohomology of the sheaf of algebraic K-groups on $Spec(R)$.)

Of these, (5.2.1) is an immediate consequence of the definitions, and (5.2.2) is a straightforward application of results in [BF]. (5.2.3) is somewhat deeper and may require some additional hypotheses on R. It is true for a large class of rings that includes all fields (see proof in [SF]) and probably all regular rings.

5.3. Motivic Cohomology? An optimist, comparing (5.2.1) with (2.2.2), might conjecture that motivic cohomology can be defined by setting

$$H^n_{\mathcal{M}}(R, j) = KV_{2j-n}^{j/j+1}(R) \tag{5.3.1}$$

(supplanting the conjecture (3.1.2)). In fact, the edge map from the spectral sequence (5.2.2) provides a homomorphism

$$KV_{2j-n}^{j/j+1}(R) \to Ch^j(X, 2j-n),$$

suggesting that the new conjecture might not differ too radically from the old conjecture (3.1.2).

Indeed, the existence of an Atiyah-Hirzebruch spectral sequence is one of the chief requisites for a motivic cohomology theory, and with the definition (5.3.1), the spectral sequence (5.2.1) fills the bill. Unfortunately, (5.3.1) fails badly as a definition in another direction. Beilinson's conjectures require that

$$H^n_{\mathcal{M}}(R,j) = 0 \quad \text{for} \quad n < 0; \tag{5.3.2}$$

his deepest ideas relate motivic cohomology to the values of L-functions in ways that would apparently be unsalvageable in the absence of (5.3.2). (For this unsalvageability I rely on the testimony of others; I am no expert on L-functions.) In particular, (5.3.1) and (5.3.2) imply that $KV_3^{0/1}(R) = H^{-3}_{\mathcal{M}}(R,0) = 0$, whereas a calculation in [SF] shows that for a field k there is an exact sequence

$$K_3^M(k) \quad \xrightarrow{i} \quad K_3(k) \quad \to \quad KV_3^{0/1}(k) \quad \to 0$$

(where K^M denotes Milnor K- theory) and the cokernel of i is often non-zero (for example, if k is any finite field).

Thus the conjecture (5.3.1) is not quite right. Whether anything like it is right remains to be seen.

5.4. Motivic Complexes? Here is another way in which the invariants of this section might be related to a motivic construction. We saw in Section 2.4 that there are conjectured to be complexes $\mathbf{Z(j)}^\bullet$, such that motivic cohomology can be defined by

$$H^n_{\mathcal{M}}(R,j) = Ext^n_{\mathcal{H}}(\mathbf{Z}(0)^\bullet, \mathbf{Z}(j)^\bullet) \tag{2.4.1}$$

These complexes $\mathbf{Z}(j)^\bullet$ are to live in some mysterious abelian category $\mathcal{H}$, not yet identified.

It will be convenient for us to introduce reindexed complexes $\tilde{\mathbf{Z}}(j)_\bullet$ defined by

$$\tilde{\mathbf{Z}}(j)_n = \mathbf{Z}(j)^{2j-n}$$

and to restate the conjecture (2.4.1) in the form

$$H^n_{\mathcal{M}}(R,j) = Ext^{n-2j}(\tilde{\mathbf{Z}}(0), \tilde{\mathbf{Z}}(j)) \tag{5.4.1}$$

Suppose that one weakens the conjecture by allowing the $\tilde{\mathbf{Z}}(j)$ to live in a category that is not quite abelian. A candidate for that category is the category of simplicial spaces (or better, the category of simplicial spectra; the "spaces" we have been dealing with are really spectra in the sense of [A]), and a candidate for $\tilde{\mathbf{Z}}(j)_\bullet$ is the simplicial spectrum $K^{j/j+1}(R,\bullet)$. (When one moves from an abelian category to a more general category, simplicial objects are the analogue of complexes.)

The general formalism of derived categories and homotopical algebra suggests that in this context, the right interpretation of $Ext^n(X,Y)$ is the set of homotopy classes of maps from X to the n^{th} topological suspension of Y. Thus (5.4.1) is converted to the provisional definition

$$H^n_{\mathcal{M}}(R,j) = [K^{0/1}(R,\bullet), \Sigma^{n-2j} K^{j/j+1}(R,\bullet)]$$

where square brackets denote homotopy classes of maps and Σ is the suspension operator.

In [SF], I construct natural maps

$$[K^{0/1}(R,\bullet), \Sigma^{n-2j} K^{j/j+1}(R,\bullet)] \to KV^{j/j+1}_{2j-n}(R) \to Ch^j(R, 2j-n),$$

thus relating the three groups that have been suggested in this paper as conjectural definitions for $H^n_{\mathcal{M}}(R,j)$.

APPENDIX: RELATIVE K-THEORY

I will use this appendix to report briefly on some related research.

Let R be a regular ring and $\{P_1,\ldots,P_m\}$ a family of prime ideals in R. Suppose always that all rings of the form $R/(\sum_I P_i)$ are regular. (I is an arbitrary subset of $\{1,\ldots,m\}$.) Then a generalization of Quillen's argument in [Q] yields a "Gersten-Quillen" spectral sequence converging to the multiply relative K-theory $K_*(R; P_1,\ldots,P_m)$. The most natural formulation of this spectral sequence uses the K-theory of categories of perfect complexes as in Section 5 above.

Consider the ring $R^{(n)}$ of Section 1.1, and consider the Gersten-Quillen spectral sequence converging to

$$K_*(R^{(n)}; (t_0), \ldots, (t_n)).$$

The terms $E_r^{p,-p}$ of this spectral sequence are of particular interest. One hopes to prove that

$$E_1^{p,-p} = Z^p(R,n) \tag{A.1}$$

and

$$E_2^{p,-p} = Ch^p(R,n). \tag{A.2}$$

These relationships would link $Ch^p(R,n)$ to $K_0(R^{(n)})$ (and consequently to $K_n(R)$) in a very natural way, and could be the key to establishing the all-important (3.1.1).

(A.1) and (A.2) are formulated for the particular ring $R^{(n)}$ and the particular family of ideals $\{(t_0),\ldots,(t_n)\}$, but it is natural to formulate them more generally. Thus if R is any regular ring and $\{P_1,\ldots,P_m\}$ any family of primes, we can study the $E_1^{p,-p}$ terms of the corresponding Gersten-Quillen spectral sequence, and make the following conjecture:

A.3. *$E_1^{p,-p}$ is a free abelian group on classes $[R/Q]$ where Q is a prime ideal that is proper of codimension p in R.*

In this case, "proper" means that for any $I \subset \{1,\ldots,m\}$, the $R/(\sum_I P_i)$-module $R/(Q+\sum_I P_i)$ always has codimension $\geq p$.

A.3. is true for $m=0$ by [Q]. It is true for $m=1$ when $P=P_1$ is principal by either [Lev] or [RCG]. It is true for $m=1$ in general by [OSS]. It is true for $m=2$ and R 2-dimensional by unpublished joint work that I have done with Rick Miranda. Bloch and Lichtenbaum have worked on related problems and may have additional results in low dimensions, though I have not seen these. However, in more than 3 or 4 dimensions the problem seems difficult.

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