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Axioms for Optimal Population

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1. Introduction

Policy decisions frequently affect population size, either as a primary goal or as a secondary consequence. This limits the applicability of approaches to social welfare which take population size as given.

In this note I will write down a list of axioms for a social welfare function that takes as arguments the utilities of an arbitrary (and variable) number of agents. I will then prove that, when those axioms hold, every social welfare function must be a monotonic transformation of one that is additively separable in the agents' utilities.

In fact, the result is a bit stronger than that; I will show that if f is a social welfare function satisfying the axioms, then there exists a monotonic function g of a single variable such that for any n and any $(x_1, \ldots, x_n)$, we have

$$g \circ f(x_1, \ldots, x_n) = \sum_{i=1}^{n} g(x_i) \tag{1.1}$$

The aims of this paper are very modest. I do not claim to have the "right" list of axioms and I do not claim that the theorem has important policy implications. I aim only to illustrate a framework for dealing with welfare when the number of agents is a variable, and to demonstrate that even with an extremely simple set of axioms, it is possible to prove at least one theorem that is not entirely trivial.

2. Axioms and Examples

2.1. The Basic Setup

I assume that there is a finite but arbitrary population of individuals, each of whom has a (cardinal) utility represented by a single real number. I assume that there is some utility level—which, after a rescaling—we can take to be zero—such that the planner is always indifferent about adding a new individual with that utility level.

Thus the social welfare function f should take a finite but arbitrary collection of real numbers as inputs, and for any vector u it should satisfy

$$f(u) = f(u, 0) \tag{2.1.1}$$

To formalize this, think of $\mathbf{R}^n$ as a subset of $\mathbf{R}^{n+1}$ by identifying the point $(x_1, \ldots, x_n) \in \mathbf{R}^n$ with the point $(x_1, \ldots, x_n, 0) \in \mathbf{R}^{n+1}$. Then let $\mathbf{X}$ be the union over n of all spaces $\mathbf{R}^n$. (Mathematicians would write $\mathbf{X} = \overset{\lim}{\rightarrow}\mathbf{R}^n$.)

A typical point in $\mathbf{X}$ is represented by an n-tuple of real numbers (n arbitrary), and the two vectors $(x_1, \ldots, x_n)$ and $(x_1, \ldots, x_n, 0)$ represent *exactly the same poiint* of X. Thus for functions defined on $\mathbf{X}$, (2.1.1) holds *by definition*.

I will abuse notation in minor and obvious ways; for example, if u and v are vectors of lengths m and n, I will write (u, v) for the vector of length $m + n$ that results from appending v to u.

For a real valued function f on $\mathbf{X}$ (or appropriate subsets thereof), it makes sense to talk about the partial derivatives of f with respect to any of the coordinates x_i; to take a partial with respect to x_n just restrict f to the subset $\mathbf{R}^n \subset X$ and use the usual definition of partial derivative for a function on $\mathbf{R}^n$.

Social welfare functions will be real-valued functions defined on subsets of $\mathbf{X}$.

2.2. Axioms for the social welfare function

Let $f : \mathbf{X} \to \mathbf{R}$ be a function. (More generally, we can take the domain to be any subset of $\mathbf{X}$ on which the axioms below continue to make sense). f is called a *social welfare function* if it satisfies the following assumptions:

Assumption 2.2.1 (Differentiability and the Pareto property) *f has first par-*

tial derivatives in every direction, and they are all positive.

Assumption 2.2.2. (Symmetry) *f is invariant under any permutation of the coordinates.*

Assumption 2.2.3 (Respect for Individual Judgments) *If x is a single real number, then $f(x) = x$.* In other words, when the population consists of a single person, the planner uses that person's utility as the measure of social welfare.

Assumption 2.2.4 (Independence Axiom) *Let u, v, and w be vectors of finite length. Then $f(v) \geq f(w)$ if and only if $f(u, v) \geq f(u, w)$.*

2.3. Theorem. *Let $\{u_i\}$ be a finite collection of vectors of finite length. Let v be another vector of finite (possibly zero) length. Then $f(u_1, \ldots, u_n, v) = f(f(u_1), \ldots, f(u_n), v)$.*

Proof. By 2.2.3, $f(u_1) = f(f(u_1))$. Thus by 2.2.4, $f(u_1, u_2, \ldots, u_n, v) = f(f(u_1), u_2, \ldots, u_n, v)$. Now apply the same argument successively to $u_2, \ldots, u_n$.

q.e.d.

2.4. Examples.

The following examples are all easily verified to be social welfare functions, except that some fail to satisfy the Pareto property in certain regions. In those cases, we restrict the domain to that region where the Pareto property holds.

2.4.1. Let

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$$

2.4.2. Given $p > 0$, let

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

.

2.4.3. Let

$$f(x_1, \ldots x_n) = \left(\prod_{i=1}^{n} (x_i + 1) \right) - 1$$

2.4.4. Given a vector $x = (x_1, \dots, x_n)$, let $S_m(x)$ be the sum of all products of m distinct elements of x. In other words,

$$S_1(x) = \sum_{i=1}^{n} x_i \qquad S_2(x) = \sum_{i \neq j} x_i x_j \qquad \text{etc.}$$

Then let

$$f(x) = \frac{S_1(x) + S_3(x) + \dots}{1 + S_2(x) + S_4(x) + \dots}$$

2.4.5. Let g be any real-valued function of a single variable which is both differentiable and monotonically increasing. Then let

$$f(x_1, \dots, x_n) = g^{-1}\left(\sum_{i=1}^{n} g(x_i)\right).$$

2.4.5.1. Special cases. With $g(x) = x$, we recover 2.4.1. With $g(x) = x^p$ we recover 2.4.2. With $g(x) = \log(x+1)$ we recover 2.4.3. With $g(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ we recover 2.4.4.

3. Results.

The main result (Corollary 3.3) is that every social welfare function is of the form given in example 2.4.5.

3.1. We begin by setting up correspondences between the set of social welfare functions and the set of functions of a single variable.

3.1.1. From social welfare functions to functions of a single variable. Given a social welfare function f, write f_1 for the partial derivative of f with respect to the first variable. Let $g = L(f)$ be the function of a single variable defined by the following conditions:

$$g'(y) = \frac{1}{f_1(0, y)} \tag{3.1.1.1}$$

$$g(0) = 0 \tag{3.1.1.2}$$

Clearly g is monotonoically increasing and differentiable. We take it to be defined on the largest possible subset of $\mathbf{R}$ such that the above conditions make sense. (Note that when f has a restricted domain, there may be values of y for which $f_1(0, y)$ is not defined).

3.1.1.3. Remark. Although (3.1.1.1) appears to give a special role to the variable x_1, the symmetry assumption (2.2.2) implies that the appearance is illusory.

3.1.2. From functions of a single variable to social welfare functions.

Let g be a real-valued differentiable monotonically increasing function defined on some subset of $\mathbf{R}$. Then define a social welfare function $f = M(g)$ (on the largest subset of $\mathbf{X}$ where this makes sense) by the condition

$$f(x_1, \ldots, x_n) = g^{-1}\left(\sum_{i=1}^{n} g(x_i)\right). \tag{3.1.2.1}$$

3.2. Theorem. *(a) For any f, $M(L(f)) = f$. (b) For any g, $L(M(g))$ is a constant multiple of g.*

3.3. Corollary. *Every social welfare function is as in Example 2.3.5. Two different functions g_1 and g_2 define the same social welfare function if and only if g_1 is a constant multiple of g_2.*

3.4. Proof of 3.2.

3.4.1. Proof of 3.2(a). Given a social welfare function f, set $g = L(f)$ (as defined in 3.1.1). We need to show that

$$(g \circ f)(x_1, \ldots, x_n) = \sum_{i=1}^{n} g(x_i). \tag{3.4.1.1}$$

As both sides of 3.4.1.1. vanish at the origin, it suffices to show that both sides have identical partial derivatives with respect to each variable. In view of remark 3.1.1.3, it suffices to show this for any particular variable. Taking that particular variable to be x_1, we need to show that

$$g'(f(x_1,\ldots,x_n))f_1(x_1,\ldots,x_n) = g'(x_1). \tag{3.4.1.2}$$

From (3.1.1.1) we can rewrite this as

$$\frac{f_1(x_1,\ldots,x_n)}{f_1(0,f(x_1,\ldots,x_n))} = \frac{1}{f_1(0,x_1)} \tag{3.4.1.3}$$

or

$$f_1(0,f(x_1,\ldots,x_n)) = f_1(x_1,\ldots,x_n)\cdot f_1(0,x_1). \tag{3.4.1.4}$$

Therefore the theorem will be proved if we can establish equation (3.4.1.4).

3.4.1.5. Proof of 3.4.1.4. Let x_0 be arbitrary. By Theorem 2.3, we have

$$f(x_0,f(x_1,\ldots,x_n)) = f(x_0,\ldots,x_n) = f(f(x_0,x_1),x_2,\ldots,x_n) \tag{3.4.1.5.1}$$

Differentiating the left and right sides of this equation with respect to x_0 and then setting $x_0 = 0$, we get

$$f_1(0,f(x_1,\ldots,x_n)) = f_1(f(0,x_1),x_2,\ldots,x_n)\cdot f_1(0,x_1). \tag{3.4.1.5.2}$$

(3.4.1.5.2) will look exactly like (3.4.1.4) if $f(0,x_1) = x_1$. But $f(0,x_1) = f(x_1,0)$ by Assumption 2.2.2 and $f(x_1,0) = f(x_1) = x_1$ by 2.1.1 and Assumption 2.2.3.

q.e.d.

3.4.2. Proof of 3.2(b). Given g, let $f = M(g)$. Then, using the explicit formula for $M(g)$ in 3.1.2.1, it is straightforward to compute that

$$\frac{1}{f_1(0,x)} = \frac{g'(x)}{g'(0)}.$$

Thus $L(M(g)) = L(f) = g/g'(0)$.

q.e.d.