## Stirling Numbers and Polylogarithms <br> Steven E. Landsburg

This note contains one (very simple) theorem about the coefficients of power series which has, as corollaries, some formulas relating polylogarithms to Stirling numbers of the second kind. (These are the less interesting polylogarithms, i.e. those of the form $L i_{-m}$ where $-m$ is a negative integer.)

I have seen other formulas relating polylogarithms to Stirling numbers but these seem not obviously equivalent to those. Nevertheless, I would not be surprised to learn that the formulas of the current paper are either a) well known or b) of no particular interest. (That is, of course, an inclusive or.)

Theorem. Let $f$ be a function with power series representations around 0 and 1:

$$
f(x)=\sum_{k=0}^{\infty} B_{k} x^{k}=\sum_{k=0}^{\infty} C_{k}(x-1)^{k}
$$

Assume the first representation is valid on a domain containing 1. Then for each $m=$ $1,2, \ldots$ (and assuming convergence) we have

$$
\sum_{k=1}^{\infty} B_{k} k^{m}=\sum_{j=1}^{m} A_{m j} C_{j} j!
$$

where the $A_{m j}$ are Stirling numbers of the second kind.
Proof. Put

$$
f_{1}=f \quad f_{m}=x f_{m-1}^{\prime}
$$

Then an induction gives

$$
f_{m}^{\prime}=\sum_{j=1}^{m} A_{m, j} x^{j-1} f^{(j)}
$$

Inserting $x=1$ gives

$$
\sum_{k=1}^{\infty} B_{k} k^{m}=f_{m}^{\prime}(1)=\sum_{j=1}^{m} A_{m, j} f^{(j)}(1)=\sum_{j=1}^{m} A_{m, j} C_{j} j!
$$

Application 1. Take $f(x)=1 /(1-n x)$. Then we get for each $m>0$ that

$$
L i_{-m}(n)=\sum_{j=1}^{m} A_{m j} \frac{n^{j}}{(1-n)^{1+j}} j!
$$

where $L i$ is the polylogarithm.

Remarks. I have seen other expressions relating polylogarithms to Stirling numbers (e.g. Wikipedia points to a 1992 technical report by David C. Wood of the University of Kent computing laboratory), but they don't seem obviously equivalent to the above.

Application 2. Take $f(x)=e^{x}$. Then we get

$$
\sum_{k=1}^{\infty} \frac{k^{m}}{k!}=e \sum_{j=1}^{m} A_{m j}
$$

Remark. This is a 19th century result known as Dobinski's Formula.
Application 2A. More generally, take $f(x)=e^{n x}$. Then (for fixed $n$ and $m$ ) we get

$$
\sum_{k=1}^{\infty} \frac{n^{k} k^{m}}{k!}=e^{n} \sum_{j=1}^{m} n^{j} A_{m j}
$$

Application 3. Take $f(x)=\cos (\pi x)$. Then we get

$$
2^{m} \sum_{k=1}^{\infty} \frac{(-1)^{k} \pi^{2 k} k^{m}}{(2 k)!}=\sum_{j=1}^{\left[\frac{m}{2}\right]}(-1)^{j+1} A_{m, 2 j} \pi^{2 j}
$$

(Here the [] indicates the floor function.)
Remark. Of course one gets a similar formula using the sine instead of the cosine.
Application 4. Take $f(x)=\cos (\pi x / 2)$. Then

$$
2^{m} \sum_{k=1}^{\infty} \frac{(-1)^{k}\left(\pi^{2 k}\right) k^{m}}{4^{k}(2 k)!}=\sum_{j=1}^{\left[\frac{m}{2}\right]} \frac{(-1)^{j} A_{m, 2 j-1} \pi^{2 j-1}}{2^{2 j-1}}
$$

Applications 5 and 6 (more polylogarithms). Taking first $f(x)=x /(1-n x)^{2}$ and then $f(x)=\log (1+n x)$ gives

$$
\begin{gathered}
L i_{-1-m}(n)=\sum_{j=1}^{m} A_{m j}(j+n) \frac{n^{j-1}}{(1-n)^{j+2}} j! \\
L i_{1-m}(-n)=\sum_{j=1}^{m}(-1)^{j} A_{m j}\left(\frac{n}{1+n}\right)^{j}(j-1)!
\end{gathered}
$$

"Application" 7. We apply the theorem to $f(x)=\log (1+x)$, even though the hypotheses fail. We get

$$
\sum_{k=1}^{\infty}(-1)^{k+1} k^{m-1}=\sum_{j=1}^{m}(-1)^{j+1} A_{m j} 2^{-j}(j-1)!
$$

Note that the right side makes sense only when $m$ is a positive integer, and that the left side never makes sense when $m$ is a positive integer; thus there is no case in which the equation is meaningful. But if we reinterpret the left side as the value of a zeta function we get

$$
(1-2 m) \zeta(1-m)=\sum_{j=1}^{m}(-1)^{j+1} A_{m j} 2^{-j}(j-1)!
$$

and this is a true formula, which suggests that something a little deeper is going on.

