II. Topology

1. Manifolds

Definition 1.1. Let M be a set. An *n-dimensional chart* on M is a pair (U, ϕ) where $U \subset M$ is a subset and $\phi : U \to \mathbf{R}^n$ is a one-one function onto an open subset $\phi(U) \subset \mathbf{R}^n$.

Think of a chart ϕ as labeling each point $p \in U$ with the n coordinates of $\phi(p) \in \mathbf{R}^n$.

Definition 1.2. Given two charts $\phi_U : U \to \mathbf{R}^n$ and $\phi_V : V \to \mathbf{R}^n$, set

$$\Omega = \phi_U(U \cap V) \subset \mathbf{R}^n \qquad \Omega' = \phi_V(U \cap V) \subset \mathbf{R}^n$$

We say that the charts ϕ_U and ϕ_V are *compatible* if

i) Ω and Ω' are both open in $\mathbf{R}^n$ and

ii) The map from Ω to Ω' defined by

$$x \mapsto \phi_V(\phi_U^{-1}(x)) \tag{1.2.1}$$

is infinitely differentiable.

("Infinitely differentiable" is to be understood in the sense of advanced calculus; note that the domain and codomain of (1.2.1) are both open subsets of $\mathbf{R}^n$.)

Definition 1.3. An *n- dimensional manifold* is a set M together with a family $\mathcal{A}$ of charts $\{(U_\alpha, \phi_\alpha)\}$ such that

i) M is the union of the U_α

ii) Every pair of charts in $\mathcal{F}$ is compatible.

iii) If (V, ψ) is a chart compatible with all the charts in $\mathcal{F}$, then $(V, \psi) \in \mathcal{A}$.

The family of charts $\mathcal{A}$ is called a *smooth structure* or a *maximal atlas* for M. (The adjective "maximal" captures the essence of condition iii). A subset U of M is called a *coordinate patch* if there exists a map ϕ such that $(U, \phi) \in \mathcal{A}$.

Remarks 1.3.1. Think of $\mathcal{A}$ as the set of allowable ways to coordinatize pieces of M. Condition i) says that every point in M can be coordinatized in at least one way.

Condition ii) says that every allowable change of coordinates is infinitely differentiable. Condition iii) says that any change of coordinates is allowable, provided it is compatible with the changes of coordinates that have already been allowed.

Abuse of Language 1.3.2. We will often call a set M a "manifold", although strictly speaking the manifold structure on M includes a specification of the family of charts $\mathcal{A}$. Whenever we speak of a manifold, we are assuming that a family of charts has been specified.

Abuse of Language 1.3.3. A family of charts that satisfies 1.3(i) and 1.3(ii) (but not necessarily 1.3(iii)) is called an *atlas*. By Zorn's Lemma, every atlas is contained in a maximal atlas. Sometimes, when we are given a set M and an atlas of charts on M, we will call M a "manifold", implicitly assuming that the atlas has been replaced by a maximal atlas that contains it.

Exercise 1.3.3.1. Prove that the maximal atlas containing a given atlas is unique, so the replacement of an atlas by a maximal atlas in (1.3.3) is unambiguous.

Remark 1.3.4. What we've called a manifold is often called a *smooth manifold*, to distinguish it from a *topological manifold*. A topological manifold is defined similarly, except that the change of coordinate maps are required to be only continuous, not infinitely differentiable. Topological manifolds will play no role in this book.

Example 1.3.5. Let V be a vector space of dimension k and let $\mathcal{A}$ be the set of all isomorphisms $V \to \mathbf{R}^k$.

Claim 1.3.5.1. $\mathcal{A}$ is an atlas for V.

Proof. We must show that if $\lambda, \mu : V \to \mathbf{R}^k$ are isomorphisms then $\mu \circ \lambda^{-1} : \mathbf{R}^k \to \mathbf{R}^k$ is infinitely differentiable in the sense of advanced calculus. More generally:

Claim 1.3.5.1.1. Any linear transformation $\mathbf{R}^k \to \mathbf{R}^\ell$ is infinitely differentiable.

Proof. It follows from the proof of (I.2.4.1) that a linear transformation must be of the form

$$(x_1, \dots, x_k) \mapsto (\sum_{j=1}^{k} \alpha_{1j} x_j, \dots \sum_{j=1}^{k} \alpha_{\ell j} x_j)$$

for some constants α_{ij}.

Convention 1.3.5.2. Henceforth, if V is a vector space, we will use the manifold structure of (1.3.5) to treat V as a manifold.

Example 1.3.6. Let M be the unit circle $\{(x, y) | x^2 + y^2 = 1\} \subset \mathbf{R}^2$. Let $U_1 = M - \{(0, 1)\}$ and $U_2 = M - \{(0, -1)\}$. Define one- one functions

$$\psi_1 : (0, 2\pi) \to U_1 \qquad \text{and} \qquad \psi_2 : (-\pi, \pi) \to U_2$$

by

$$\psi_i(x) = (\cos(x), \sin(x))$$

and define $\phi_i : U \to \mathbf{R}^1$ by

$$\phi_i = \psi_i^{-1}$$

Then (U_1, ϕ_1) and (U_2, ϕ_2) constitute an atlas for M, which can be extended to a maximal atlas, making M a manifold.

Exercise 1.3.7. Let $M = \mathbf{S}^2$ be the 2−sphere $\{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbf{R}^3$.

Let $Z = \{(x, y, z) \in \mathbf{S}^2 | x \leq 0, y = 0\} \subset \mathbf{S}^2$ and let Ω be the complement of Z in $\mathbf{S}^2$. Map $(-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbf{R}^2$ to Ω by

$$\psi : (u, v) \mapsto \big(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)\big)$$

and let $\phi = \psi^{-1} : U \to \mathbf{R}^2$. Show that (Ω, ϕ) is a well- defined chart on $\mathbf{S}^2$. By composing ϕ with rotations of the sphere, construct additional charts to make $\mathbf{S}^2$ a manifold.

You should be able to visualize the coordinates u and v as "longitude" and "latitude".

Definition 1.4. Let M be a manifold. A subset $U \subset M$ is *open in M* (or just *open* for short) if it is a union of coordinate patches.

1.4.1. If $U \subset M$ is open, then there is a natural way to make U into a manifold: For every chart (U_α, ϕ_α) on M, declare $(U \cap U_\alpha, \phi_\alpha|_{U \cap U_\alpha})$ to be a chart on U.

Exercises 1.5. Let M be a manifold with maximal atlas $\mathcal{A}$. Prove the following:

a) A subset $U \subset M$ is open if and only if: for every $(U_\alpha, \phi_\alpha) \in \mathcal{A}$, the set $\phi_\alpha(U \cap U_\alpha)$ is open in $\mathbf{R}^n$.

b) If $(U, \phi) \in \mathcal{A}$ and $V \subset U$ is a subset, then V is open in M if and only $\phi(V)$ is open in $\mathbf{R}^n$.

c) If $(U, \phi) \in \mathcal{A}$ and $V \subset U$ is open, then $(V, \phi|_V) \in \mathcal{A}$. (Use the maximality condition (1.3.iii).)

d) The union of any number of open sets is open. The intersection of finitely many open sets is open.

Definition 1.6. Let M and M' be manfolds with maximal atlases $\mathcal{A}$ and $\mathcal{A}'$. A function $f : M \to M'$ is called *smooth* if for every $(V, \psi) \in \mathcal{A}'$, there is a subset $\{U_\alpha, \phi_\alpha)\} \subset \mathcal{A}$ such that

i) $f^{-1}(V) = \bigcup_\alpha U_\alpha$

ii) For each α, the map

$$\psi \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \psi(V) \tag{1.6.1}$$

is infinitely differentiable.

Exercise 1.6.2. Show that a composition of smooth maps is smooth.

Exercise 1.6.3. Let $\mathcal{B}' \subset \mathcal{A}'$ be a (non-maximal) atlas on M'. Suppose that for every $(V, \psi) \in \mathcal{B}'$, there is a subset $\{(U_\alpha, \phi_\alpha)\} \subset \mathcal{A}$ such that (1.6i) and (1.6ii) hold. Show that f is smooth. Therefore, verifying smoothness can be much easier than you might think from a naive reading of (1.6).

Example 1.6.4. Let V and W be vector spaces, thought of as manifolds via (1.3.5.) Then any linear transformation $f : V \to W$ is smooth.

Proof. Choose isomorphisms $\phi : V \to \mathbf{R}^k$ and $\psi : W \to \mathbf{R}^\ell$ (these exist for some k and ℓ by (I.1.10). Let $\mathcal{B}'$ be the singleton set $\{(W, \psi)\}$, which is an atlas for W. Then the singleton $\{(V, \phi)\}$ satisfies (1.6i) and (1.6ii), so by (1.6.2) it suffices to check that $\psi \circ f \circ \phi^{-1}$

is infinitely differentiable, which follows from (1.3.5.1.1).

Convention 1.6.5. Henceforth, if we write down a map from one manifold to another, the map is assumed to be smooth.

1.7. In (1.8) we will record some additional facts about smooth maps between vector spaces for later reference. First we need:

Lemma 1.7.1. Let $\Omega \subset \mathbf{R}^m$ be an open subset and let $g : \Omega \to \mathbf{R}^k$ be a (not necessarily differentiable) function. Let $x_i : \mathbf{R}^k \to \mathbf{R}$ be the i^{th} coordinate function and let $g_i = x_i \circ g : \Omega \to \mathbf{R}$. Then the following statements are equivalent:

i) The map $g : \Omega \to \mathbf{R}^k$ is infinitely differentiable.

ii) Each of the maps $g_i : \Omega \to \mathbf{R}$ is infinitely differentiable.

iii) The map

$$\begin{array}{ccc} \Omega \times \mathbf{R}^k & \to & \mathbf{R} \\ (\beta_1, \dots \beta_m, \alpha_1, \dots, \alpha_k) & \mapsto & \sum_{i=1}^k \alpha_i g_i(\beta_1, \dots, \beta_m) \end{array}$$

is infinitely differentiable.

Proof. This is a straightforward exercise in advanced calculus.

Proposition 1.8. Let M be a manifold, V a vector space, $\{v_1, \dots, v_n\}$ a basis for V, and $f : M \to V$ a (not necessarily smooth) function. For each $m \in M$, write (uniquely)

$$f(m) = \sum_{i=1}^{n} f_i(m) v_i$$

with $f_i(m) \in \mathbf{R}$. Then the following statements are equivalent:

i) The map $f : M \to V$ is smooth.

ii) Each of the maps $f_i : M \to \mathbf{R}$ is smooth.

iii) The map

$$\begin{array}{ccc} M \times \mathbf{R}^k & \to & \mathbf{R} \\ (m, \alpha_1, \dots, \alpha_k) & \mapsto & \sum_{i=1}^k \alpha_i f_i(m) \end{array}$$

is smooth.

Proof. Define a linear transformation $\psi : V \to \mathbf{R}^n$ by

$$\psi\left(\sum_{i=1}^{n} \alpha_i v_i\right) = (\alpha_1, \ldots, \alpha_n)$$

Note that (V, ψ) is a chart on V.

Given a chart (U, ϕ) on M, set $\Omega = \phi(U) \subset \mathbf{R}^m$, set $g = \psi \circ f \circ \phi^{-1} : \Omega \to \mathbf{R}^k$, and consider the statements

i′) (1.7.1i) holds for every g that arises in this way.

ii′ (1.7.1ii) holds for every g that arises in this way.

iii′) (1.7.1iii) holds for every g that arises in this way.

By (1.6.3), (i), (ii) and (iii) are equivalent to (i′), (ii′) and (iii′) and hence, by (1.7.1), equivalent to each other.

Definition 1.9. A smooth bijection with a smooth inverse is called a *diffeomorphism*.

Definition 1.10. Let M_1 and M_2 be manifolds, let V_1 and V_2 be open subsets of M_1 and M_2, and let $f : V_1 \to V_2$ be a diffeomorphism. We define a new manifold, $M_1 \cup_f M_2$ called the *patching* of M_1 and M_2 along f, as follows:

As a set, $M = M_1 \cup_f M_2$ is the disjoint union of M_1 and M_2, modulo the equivalence relation $v \sim f(v)$ for all $v \in V_1$. In order to make M a manifold, we still have to define charts.

For $i = 1, 2$ we use the obvious injection $M_i \hookrightarrow M$ to identify M_i with its image. Thus each subset of M_i is identified with a subset of M, so each chart on M_i gives a chart on M. These charts (expanded to a maximal atlas per (1.3.3) make M a manifold.)

Exercise 1.10.1. Explicitly describe the unit circle as a patching of the sets U_1 and U_2 from (1.3.6).

Exercise 1.10.2. Explicitly describe the 2-sphere as a patching of two subsets.

Definition 1.11. Let M_1 and M_2 be manifolds of dimension n_1 and n_2. We define the $(n_1 + n_2)$-dimensional *product manifold* $M_1 \times M_2$ to be the cartesian product of the sets M_1 and M_2 provided with charts as follows:

Given charts (U_1, ϕ_1) on M_1 and (U_2, M_2) on M_2, define a chart $(U_1 \times U_2, \phi)$ on $M_1 \times M_2$ by

$$(u_1, u_2) \to (\phi_1(u_1), \phi_2(u_2)) \in \mathbf{R}^{n_1+n_2}$$

As the (U_i, ϕ_i) range over charts for the M_i, these charts form an atlas for $M_1 \times M_2$. Extending this atlas to a maximal atlas, $M_1 \times M_2$ becomes a manifold.

Exercise 1.11.1. Show that a map to a product manifold $M_1 \times M_2$ is smooth if and only if it becomes smooth after composing with each of the projection maps $M_1 \times M_2 \to M_i$.

2. Vector Bundles

2.1. Intuition. Intuitively, a *vector bundle* over a manifold M is a manifold $\mathcal{E}$ that can be constructed by attaching a vector space $\mathcal{E}_m$ to each point $m \in M$ in a smooth way. For example, let $M = \mathbf{R}$ (thought of as a manifold) and attach to each point of M a copy of $\mathbf{R}$ (thought of as a one-dimensional vector space). Imagine M as a horizontal line and the attached vector spaces as vertical lines, and you'll see that you've constructed $\mathbf{R}^2$. Thus $\mathbf{R}^2$ is a vector bundle over $\mathbf{R}$.

For another example, let $\mathbf{S}^1$ be the unit circle. Attach a vertical line (a one- dimensional vector space) to each point of $\mathbf{S}^1$ and you'll get a cylinder; thus the cylinder is a vector bundle over $\mathbf{S}^1$. For a more interesting example, imagine giving each line a "twist" as you attach it, slowly increasing the angle of the twist until you come back around to the starting point, at which point you've twisted a full 180 degrees. (To prevent the twisted lines from getting all tangled up with each other, use short open intervals instead of infinitely long lines and identify each short interval with the full real line via a function like $x \mapsto \tan(x)$.) This constructs a Mobius strip, so a Mobius strip is another example of a vector bundle over $\mathbf{S}^1$.

A vector bundle $\mathcal{E}$ over M is a manifold in its own right, but it also comes equipped with extra structure. First, there is the map $p : \mathcal{E} \to M$ that takes $\mathcal{E}_m \subset \mathcal{E}$ to m; the pair $(\mathcal{E}, p)$ is an example of what we will call a "manifold over M". Next, there is the vector space structure on each $\mathcal{E}_m$, which makes $(\mathcal{E}, p)$ a vector bundle.

Our immediate goal is to make all of this precise. First we will study the properties of manifolds over M. Then in (2.10) we will define *trivial vector bundles*, which, intuitively, are those, like the cylinder, which can be constructed without "twisting"; that is, they look like the cartesian product of M with a vector space. Finally, in (2.15), we will define the general concept of a vector bundle.

2A. Manifolds Over M.

Definition 2.2. Let M be a manifold. A *manifold over* M is a manifold $\mathcal{E}$ together with a map $p : \mathcal{E} \to M$.

Abuse of language 2.2.1. We will sometimes call $\mathcal{E}$ a manifold over M, suppressing the reference to the map p. When we want to distinguish between $\mathcal{E}$ (which is a manifold) and $(\mathcal{E}, p)$ (which is a manifold over M), we will call $\mathcal{E}$ the *total space* of $(\mathcal{E}, p)$.

Example 2.2.2. Any manifold of the form $M \times N$ is a manifold over M by projection onto the first factor.

Definition 2.3. Let $p : \mathcal{E} \to M$ be a manifold over M and let $U \subset M$ be an open subset. Then the *restriction of* $\mathcal{E}$ *to* U is the manifold $p^{-1}(U)$, which is a manifold over U via the restriction of the map p. We write $\mathcal{E}|_U$ for the restriction of $\mathcal{E}$ to U.

Trivial Exercise 2.3.1. Let U and V be open subsets of M. Prove that

$$(\mathcal{E}|_U)\,|_V = (\mathcal{E}|_V)\,|_U = \mathcal{E}|_{U \cap V}$$

Definition 2.4. Let $p : \mathcal{E} \to M$ be a manifold over M and let $m \in M$. Then the *fiber of* $\mathcal{E}$ *over* m is the set $\mathcal{E}_m = p^{-1}(m) \subset \mathcal{E}$. Note that in general there is no obvious manifold structure on $p^{-1}(m)$. Note also that $\mathcal{E}$ is the disjoint union of the sets $\mathcal{E}_m$.

Example 2.4.1. If $\mathcal{E} = M \times N$, (2.2.2) then $\mathcal{E}_m$ can be identified with N via projection on the second factor. In this case, $\mathcal{E}_m$ does have an obvious manifold structure.

Definition 2.5. Let $f : M \to N$ be a map of manifolds. Suppose we have manifolds over M and N as indicated:

$$\begin{array}{ccc} \mathcal{E} & & \mathcal{F} \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ M & \xrightarrow{f} & N \end{array}$$

Then a map $\phi : \mathcal{E} \to \mathcal{F}$ is called a *map over* f if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\phi} & \mathcal{F} \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ M & \xrightarrow{f} & N \end{array}$$

(Commutativity means that $p_2 \circ \phi = f \circ p_1$.)

Note that the notion of a "map over f" depends on the maps p_1 and p_2, although this dependence is suppressed in the terminology.

Definition 2.6. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be manifolds over M. A map $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is called an *M-map* if it is a map over the identity map on M. An *M- diffeomorphism* is a map that is both a diffeomorphism and an M-map. We say that $\mathcal{E}_1$ and $\mathcal{E}_2$ are *M- diffeomorphic* if there is an M-diffeomorphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$.

Exercise and Notation 2.6.1. Let $f : \mathcal{E} \to \mathcal{F}$ be an M- map and let $m \in M$ be a point. Show that $f(\mathcal{E}_m) \subset \mathcal{F}_m$. We will write

$$f_m : \mathcal{E}_m \to \mathcal{F}_m \tag{2.6.1.1}$$

for the restriction of f to $\mathcal{E}_m$.

2B. Trivializations and Vector Bundles

Definition 2.7. Let $\mathcal{E}$ be a manifold over M. A *rank-k trivialization* of $\mathcal{E}$ is an M-diffeomorphism

$$\tau : \mathcal{E} \to M \times \mathbf{R}^k \tag{2.7.1}$$

A *trivialization* is a map that is a rank-k trivialization for some k.

Example 2.7.2. The identity map on $M \times \mathbf{R}^k$ is a trivialization.

Definition 2.7.3. A manifold $\mathcal{E}$ over M is *trivializable* if there exists a trivialization (2.7.1).

Remark 2.7.4. If $U \subset M$ is an open subset, then a trivialization of $\mathcal{E}$ restricts to a trivialization of $\mathcal{E}|_U$ (2.3).

Construction 2.8. Given a trivialization (2.7.1) and a point $m \in M$, we apply (2.6.1.1) to get a map

$$\tau_m : \mathcal{E}_m \to \{m\} \times \mathbf{R}^k$$

Composing with projection onto the second factor gives a map

$$\tilde{f}_m : \mathcal{E}_m \to \mathbf{R}^k$$

Because τ is a diffeomorphism, both τ_m and $\tilde{f}_m$ are one-one and onto. It therefore makes sense to define vector space operations on $\mathcal{E}_m$ by setting

$$e_1 + e_2 = \tilde{f}_m^{-1}(\tilde{f}_m(e_1) + \tilde{f}_m(e_2))$$

$$\alpha e_1 = \tilde{f}_m^{-1}(\alpha \tilde{f}_m(e))$$

Definition 2.9. Two trivializations of $\mathcal{E}$ are *equivalent* if, for every $m \in M$, they induce the same vector space structure on the fiber $\mathcal{E}_m$.

Exercise 2.9.1. Let σ and τ be rank k trivializations of $\mathcal{E}$. σ and τ are equivalent if and only if all of the maps $\tilde{\sigma}_m \circ \tilde{\tau}_m^{-1}$ are linear transformations (and hence necessarily isomorphisms) of $\mathbf{R}^k$.

Definition 2.10. A *trivial vector bundle* is a manifold $\mathcal{E}$ over M together with an equivalence class of trivializations of $\mathcal{E}$.

Remark 2.10.1. To specify an equivalence class it is enough to name one member of the class. So to specify a trivial vector bundle, it is enough to specify a manifold $\mathcal{E}$ over M and one trivialization of $\mathcal{E}$.

Example and Abuse of Language 2.10.2. $M \times \mathbf{R}^k$, together with the equivalence class of the identity map, is a trivial vector bundle. We will abbreviate this trivial vector bundle $M \times \mathbf{R}^k$, suppressing reference to the trivialization.

Definition 2.11. Let $\mathcal{E}$ be a manifold over M and let U and V be open subsets of M. Suppose we are given trivializations

$$\tau_U : \mathcal{E}|_U \to U \times \mathbf{R}^k$$

$$\tau_V : \mathcal{E}|_V \to V \times \mathbf{R}^k$$

We say that τ_U and τ_V are *compatible* if their restrictions to $U \cap V$ (2.7.4) are equivalent (2.9).

Remark 2.11.1. For $m \in U \cap V$, it is an immediate consequence of the definition that if τ_U and τ_V are compatible, then they induce the same vector space structure on $\mathcal{E}_m$.

Definition 2.12. Let $\mathcal{E}$ be a manifold over M. A *local trivialization* of $\mathcal{E}$ consists of

i) a family of open subsets $\{U_\alpha\}$ in M whose union is all of M and

ii) for each U_α, a trivialization τ_α of $\mathcal{E}|_{U\alpha}$, such that for any two indices α and β, the trivializations τ_α and τ_β are compatible (2.11).

$\mathcal{E}$ is called *locally trivializable* if there exists a local trivialization of $\mathcal{E}$.

Example 2.12.1. Any trivialization is a local trivialization. (Take the family $\{U_\alpha\rangle$ to be the singleton $\{M\}$.)

Construction 2.13. Given a local trivialization of $\mathcal{E}$ and given a point $m \in M$, we define a vector space structure on $\mathcal{E}_m$ as follows: First choose α such that $m \in U_\alpha$, then apply construction (2.8) to $\mathcal{E}|_{U_\alpha}$. The outcome is independent of the choice of α by (2.11.1).

Definition 2.14. Two local trivializations of $\mathcal{E}$ are *equivalent* if, for every $m \in M$, they induce the same vector space structure on the fiber $\mathcal{E}_m$.

Definition 2.15. A *vector bundle* is a locally trivializable manifold $\mathcal{E}$ over M together with an equivalence class of local trivializations. Given a vector bundle, we will treat each fiber $\mathcal{E}_m$ as a vector space via construction (2.13).

A vector bundle has *rank* k if all the fibers $\mathcal{E}_m$ have dimension k as vector spaces.

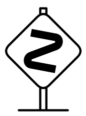
Abuse of Language 2.15.1. When a vector bundle structure on $\mathcal{E}$ has been specified, we will often say that $\mathcal{E}$ is a vector bundle, suppressing reference to the local trivializations.

Example 2.15.2. By (2.12.1), any trivial vector bundle is a vector bundle. In particular, $M \times \mathbf{R}^k$ is a vector bundle.

Definition 2.16. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles over M. A *map of vector bundles* is an M-map $f : \mathcal{E} \to \mathcal{F}$ such that for every $m \in M$, the induced map

$$f_m : \mathcal{E}_m \to \mathcal{F}_m$$

(2.6.1.1) is a linear transformation of vector spaces.



Remark 2.16.1 Not every M- map $f : \mathcal{E} \to \mathcal{F}$ is a map of vector bundles.

Definition 2.17. A map of vector bundles is an *isomorphism* if it has a two-sided inverse that is also a map of vector bundles. In particular, an isomorphism of vector bundles must be an M- diffeomorphism and it must induce isomorphisms of vector spaces on every fiber. Two vector bundles are *isomorphic* if there is an isomorphism from one to the other.

Definition 2.18. A vector bundle is *trivial* if it isomorphic to the vector bundle $M \times \mathbf{R}^k$.

Exercise 2.18.1. Show that a vector bundle is trivial (in the sense of (2.18) if and only if it is a trivial vector bundle (in the sense of (2.10).)

Our next results classify maps between trivial bundles. Proposition (2.19) and its corollaries (2.19.1) and (2.19.2) will be needed only in Section 2C (which is optional) and in the proof of (2.20.1).

Proposition 2.19. Let M be a manifold and let $\lambda : M \to \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^\ell)$ be any function (not necessarily smooth!) Define

$$\mu : M \times \mathbf{R}^k \to M \times \mathbf{R}^\ell$$

by

$$\mu(m, x) = (m, \lambda(m)(x))$$

Then λ is smooth (and hence a map of vector bundles) if and only if μ is smooth.

(Here $\mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^\ell)$ is treated as a manifold via (1.3.5.2).)

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for $\mathbf{R}^k$, $\{w_1, \ldots, w_\ell\}$ a basis for $\mathbf{R}^\ell$, and $\{f_{ij}\}$ the basis for $\mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^\ell)$ defined in the proof of (I.2.4).

There are unique functions $\lambda_{ij} : M \to \mathbf{R}$ such that

$$\lambda(m) = \sum_{i,j} \lambda_{ij}(m) f_{ij}$$

By the equivalence of (1.8i) and (1.8ii), λ is smooth if and only if all the λ_{ij} are smooth.

It's easy to compute that

$$\mu(m, v_i) = \left(m, \sum_j \lambda_{ij}(m) w_j \right)$$

By the equivalence of (1.8(ii)) and (1.8(iii)), together with (1.11.1), μ, like λ, is smooth if and only if all the λ_{ij} are smooth.

Corollary 2.19.1. Let M be a manifold. There is a one-one correspondence between

i) Smooth maps

$$\lambda : M \to \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^\ell)$$

ii) Vector bundle maps

$$\mu : M \times \mathbf{R}^k \to M \times \mathbf{R}^\ell$$

Proof. Given λ as in (i), define μ by

$$\mu(m, x) = (m, \lambda(m)(x))$$

Given μ as in (ii), write $\mu(m, x) = (m, \tilde{\mu}(m, x))$ and define λ by

$$\lambda(m)(x) = \tilde{\mu}(m, x)$$

Clearly these operations are inverse to each other.

Corollary 2.19.2. Let $GL_k(\mathbf{R}) \subset \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^k)$ be the set of isomorphisms from $\mathbf{R}^k$ to itself. Define a map to $GL_k(\mathbf{R})$ to be smooth if it is smooth as a map to $\mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^k)$. Then there is a one-one correspondence between

i) Smooth maps

$$M \to GL_k(\mathbf{R})$$

ii) Vector bundle isomorphisms

$$M \times \mathbf{R}^k \to M \times \mathbf{R}^k$$

2.20. Constructing Vector Bundles. We will describe a general procedure for constructing vector bundles over a manifold M. The technical points are conceptually simple but notationally frightening, particularly in the proof of (2.20.1). This material will be used only in the verification that definition (3.4) makes sense; the reader who is willing to gloss over a few technical points in (3.4) can safely skip the remainder of Section 2B.

Suppose we are given the following data:

i) A *set* (not a manifold!) $\mathcal{E}$

ii) A function $p : E \to M$.

iii) A family of charts $\{(U_\alpha, \phi_\alpha)\}$ on M

iv) A collection of bijections

$$\tau_\alpha : \mathcal{E}_{U_\alpha} = p^{-1}(U_\alpha) \to U_\alpha \times \mathbf{R}^k$$

We would like to give $\mathcal{E}$ the structure of a vector bundle, with the (U_α, τ_α) as a local trivialization. But before $\mathcal{E}$ can be a vector bundle, it has to be a manifold, which means we need charts for $\mathcal{E}$. The obvious candidates for coordinate patches are the pairs $(E_{U_\alpha}, \psi_\alpha)$ where ψ_α is the composition

$$\begin{array}{rccccccc} \psi_\alpha : & \mathcal{E}_{U_\alpha} & \xrightarrow{\tau_\alpha} & U_\alpha \times \mathbf{R}^k & \to & \mathbf{R}^n \times \mathbf{R}^k & \approx & \mathbf{R}^{nk} \\ & & & (m, x) & \mapsto & (\phi_\alpha(m), x) & & \end{array}$$

where we have chosen once and for all an arbitrary isomorphism $\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^{n+k}$.

In order for the ψ_α to be a compatible family of charts, it is necessary and sufficient that for every α and β, the map $\psi_\beta \circ \psi_\alpha^{-1}$ is infinitely differentiable on the domain where it is defined.

Thus: $\mathcal{E}$ (with the extra structure just described) is a vector bundle over M if and only if the $\psi_\beta \circ \psi_\alpha^{-1}$ are all infinitely differentiable. The following proposition supplies some further equivalent conditions:

Proposition 2.20.1. Given data as in (2.20), define $\tilde{\tau}_\alpha : \mathcal{E}_{U_\alpha} \to \mathbf{R}^k$ by the equation

$$\tau_\alpha(e) = \big(p(e), \tilde{\tau}_\alpha(e)\big) \in U_\alpha \times \mathbf{R}^k$$

so that

$$\psi_\alpha(e) = \big(\phi_\alpha(p(e)), \tilde{\tau}_\alpha(e)\big) \in \mathbf{R}^n \times \mathbf{R}^k$$

Write $\Omega_{\alpha\beta} = \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$. Then the following conditions are equivalent:

i) The construction of (2.20) makes $\mathcal{E}$ a vector bundle over M

ii) All of the maps

$$\psi_\beta \circ \psi_\alpha^{-1} : \Omega_{\alpha\beta} \times \mathbf{R}^k \to \Omega_{\beta\alpha} \times \mathbf{R}^k$$

are infinitely differentiable.

iii) All of the maps

$$\tau_\beta \circ \tau_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{R}^k \to (U_\alpha \cap U_\beta) \times \mathbf{R}^k$$

are smooth.

iv) All of the maps

$$\lambda_{\alpha_\beta} : (U_\alpha \cap U_\beta) \times \mathbf{R}^k \quad \to \quad \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^k)$$

defined by

$$\lambda_{\alpha\beta}(m)(x) = (\tilde{\tau}_\beta \circ \tilde{\tau}\alpha^{-1})(m,x)$$

are smooth.

Proof. The equivalence of (i) and (ii) has already been noted; (ii) just says that the ψ_α form an atlas for $\mathcal{E}$, and once $\mathcal{E}$ has the structure of a manifold, the τ_α immediately give it the structure of a vector bundle.

The equivalence of (ii) and (iii) is essentially the definition of smoothness; see (1.6) and (1.6.3).

The equivalence of (iii) and (iv) is (2.19) applied to the case $M = U_\alpha \cap U_\beta$ and $\lambda = \lambda_{\alpha\beta}$

2C. Patching.

In this section, we will describe another way to construct vector bundles—roughly, take a covering of M, construct trivial vector bundles over the sets in the covering, and patch them together by the method of (2.9). None of this material will be strictly necessary for anything that follows, so readers in a hurry can skip all of Section 2C.

Example 2.21. Let $\mathbf{S}^1$ be the unit circle in $\mathbf{R}^2$. Then $\mathbf{S}^1 \times \mathbf{R}^1$ (together with the projection map to $\mathbf{S}^1$) is a trivial rank-1 vector bundle on $\mathbf{S}^1$. Think of this vector bundle as a collection of vertical lines $\mathcal{E}_m$, one over each point m of the circle, and each endowed with the structure of a one- dimensional vector space.

To construct a non-trivial vector bundle on $\mathbf{S}^1$, we need to attach a line to each point on the circle in a way that looks locally like the construction of a cylinder (so that it will be a vector bundle) and globally like something else (so that it will be non- trivial).

To do this, cover the circle with two open sets, $U = \mathbf{S}^1 - \{(1,0)\}$ and $V = \mathbf{S}^1 - \{(-1,0)\}$. Construct trivial vector bundles $\mathcal{E}_U = U \times \mathbf{R}^1$ and $\mathcal{E}_V = V \times \mathbf{R}^1$. Let $\mathcal{E}_U|_V$ be the restriction of $\mathcal{E}_U$ to $U \cap V$ and let $\mathcal{E}_V|_U$ be the restriction of $\mathcal{E}_V$ to $U \cap V$. Map

$$\mathcal{E}_U|_V \to \mathcal{E}_V|_U$$

by

$$(u,x) \mapsto \begin{cases} (u,x) & \text{if x is in the upper half-plane} \\ (u,-x) & \text{if x is in the lower half-plane} \end{cases} \tag{2.21.1}$$

Now use (1.10) to patch $\mathcal{E}_U$ to $\mathcal{E}_V$ along this map and try to envision the result. You'll know you've understood this example when you can see that the resulting manifold is a Mobius strip, with a local trivialization given by the covering $\{U, V\}$ and the trivializations of $\mathcal{E}_U$ and $\mathcal{E}_V$ that are inherent in their definitions.

Exercise 2.21.2 With the notation of (2.20) Let $f : U \cap V \to \mathbf{R} - \{0\}$ be any smooth map, and replace (2.20.1) with

$$(u,x) \mapsto (u, f(u)x) \tag{2.21.3}$$

and show that the vector bundle you construct must be isomorphic either to the trivial bundle or the Mobius strip. What property of f distinguishes between the two possibilities?

Discussion 2.22. In general, a good way to construct vector bundles is to start with a covering, construct trivial vector bundles over each piece of the covering, and then patch. Specifically, we can take $\mathcal{E}_U = U \times \mathbf{R}^k$ and $\mathcal{E}_V = V \times \mathbf{R}^k$.

To patch the manifolds $\mathcal{E}_U$ and $\mathcal{E}_V$ and get a manifold over M, we need an M-diffeomorphism

$$\begin{array}{cccc} \phi_{UV} : & (\mathcal{E}_U)\,|_V & \longrightarrow & (\mathcal{E}_V)\,|_U \\ & \| & & \| \\ & (U \cap V) \times \mathbf{R}^k & & (U \cap V) \times \mathbf{R}^k \end{array}$$

In order for the trivializations of $\mathcal{E}_U$ and $\mathcal{E}_V$ to be compatible, ϕ_{UV} must induce linear transformations (and therefore isomorphisms) on all fibers. In other words, ϕ_{UV} must be an isomorphism from the trivial bundle $(U \cap V) \times \mathbf{R}^k$ to itself.

According to (2.19.2), choosing ϕ_{UV} is the same thing as choosing a smooth map

$$U \cap V \to GL_k(\mathbf{R}) \tag{2.22.1}$$

Thus a smooth map (2.22.1) suffices to define a vector bundle over M. (Exercise: What map of the form (2.22.1) defines the Mobius strip? Hint: the map must correspond, via (2.19.2), to the map (2.21.1).)

When there are more than two sets in the covering, it's important that the various patchings all be appropriately compatible with each other. The necessary ingredients are formalized in the following definition:

Definition 2.23. Let M be a manifold and let $\{U_\alpha\}$ be a collection of open sets whose union is M. For each pair of indices (α, β), let $f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbf{R})$ be a smooth map where $GL_k(\mathbf{R})$ is as in (2.19.2).

Suppose that for each triple of indices (α, β, γ), we have

$$f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$$

afte r all functions have been restricted to the set $U_\alpha \cap U_\beta \cap U_\gamma$ (so that both sides of the equation make sense).

Then the collection of maps $\{f_{\alpha\beta}\}$ is called a *cocycle* with values in $GL_k(\mathbf{R})$.

Proposition 2.24. To every cocycle there is an associated vector bundle, constructed by patching the various $\mathcal{E}_\alpha = U_\alpha \times \mathbf{R}^k$ along the maps induced by the $f_{\alpha\beta}$.

In detail: Consider the disjoint union of the $\mathcal{E}_\alpha$, modulo the equivalence relation

$$(u, x)_\alpha \sim (u, f_{\alpha\beta}(x))_\beta$$

for $u \in U_\alpha \cap U_\beta$ and $(u, x)_\alpha$ the copy of (u, x) that is contained in $\mathcal{E}_\alpha$. Now check that $\mathcal{E}$ is a vector bundle over M.

2.25. Proposition (2.24) says that every cocycle yields a vector bundle. It is natural to ask whether every vector bundle can be constructed in this way. The next proposition shows that the answer is yes.

Proposition 2.25.1. Let $\mathcal{E}$ be a vector bundle over M and choose a local trivialization of $\mathcal{E}$ as in (2.12). For each pair of indices (α, β), define a map

$$f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbf{R})$$

as follows: For $m \in U_\alpha \cap U_\beta$ define $f_{\alpha\beta}(m)$ to be the composition

$$\begin{array}{ccccccccc} \mathbf{R}^k & \to & (U_\alpha \cap U_\beta) \times \mathbf{R}^k & \xrightarrow{\phi_\alpha^{-1}} & \mathcal{E}|_{U_\alpha \cap U_\beta} & \xrightarrow{\phi_\beta} & (U_\alpha \cap U_\beta) \times \mathbf{R}^k & \to & \mathbf{R}^k \\ x & \mapsto & (m, x) & & & & (m, x) & \mapsto & x \end{array}$$

Then $\{f_{\alpha\beta}\}$ is a cocycle, and the associated vector bundle is isomorphic to $\mathcal{E}$.

2D. Sections

Definition 2.26. Let $p : \mathcal{E} \to M$ be a vector bundle. A *smooth section* of $\mathcal{E}$ (or just a *section* for short) is a smooth map $s : M \to \mathcal{E}$ such that

$$p \circ s = 1_M$$

We write $\Gamma(M, \mathcal{E})$ for the set of all sections $s : M \to \mathcal{E}$. If $U \subset M$ is an open subset, we write $\Gamma(U, \mathcal{E})$ for $\Gamma(U, \mathcal{E}|_U)$.

Exercise 2.26.1. If $\mathcal{E}$ is a trivial bundle, show that every element of $\mathcal{E}$ is in the image of some smooth section.

Notation 2.27. Write $\mathcal{C}(M)$ for the set of all smooth real-valued functions on M. Let U be an open subset of M. For $s, t \in \Gamma(U, \mathcal{E})$, define the sum $s + t$ by

$$(s + t)(u) = s(u) + t(u)$$

For $\phi \in \mathcal{C}(M)$ and $s \in \Gamma(U, \mathcal{E})$, define the product $\phi s \in \Gamma(U, \mathcal{E})$ by

$$(\phi s)(u) = \phi(u)s(u)$$

Remark 2.28. The addition and multiplication rules of (2.27) satisfy the analogues of the vector space axioms in (I.1.1). We summarize this situation by saying that $\Gamma(U, \mathcal{E})$ is a *module* over $\mathcal{C}(M)$.

Definition 2.29. A family of sections $\{s_1, \ldots, s_k\}$ is a *global basis* for $\mathcal{E}$ if for every $m \in M$, $\{s_1(m), \ldots, s_k(m)\}$ is a basis for $\mathcal{E}_m$.

Proposition 2.30. Let $\mathcal{E}$ be a vector bundle of rank k over M. Then $\mathcal{E}$ has a global basis if and only if $\mathcal{E}$ is trivial.

Proof. First, suppose $\mathcal{E}$ has a global basis $s_1, \ldots, s_k$.

Let $e_1 \ldots e_k$ be a basis for $\mathbf{R}^k$. and map

$$M \times \mathbf{R}^k \to \mathcal{E}$$

by

$$\left(m, \sum \alpha_i e_i\right) \mapsto \left(\sum \alpha_i s_i(m)\right)$$

Check that this map is an isomorphism. (Use (I.1.17.2).)

Conversely, suppose $\mathcal{E}$ is trivial. Choose an isomorphism

$$f : M \times \mathbf{R}^k \to \mathcal{E}$$

Define $s_i : M \to \mathcal{E}$ by $s_i(m) = f(m, e_i)$, and check that the s_i form a global basis.

Corollary 2.30.1. Let $s_1, \ldots, s_k$ be a global basis for $\mathcal{E}$. Then every smooth section of $\mathcal{E}$ can be written uniquely as

$$\sum_{i=1}^{k} a_i s_i$$

with $a_i \in \mathcal{C}(M)$.

Remarks 2.31. Let $f : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles. Then for any open set U, f induces a map of sections

$$\begin{array}{rccc} \tilde{f}_U : & \Gamma(U, \mathcal{E}) & \to & \Gamma(U, \mathcal{F}) \\ & s & \mapsto & f|_{\mathcal{E}|_U} \circ s \end{array}$$

These maps "fit together" in the sense that for open sets $V \subset U$, and $s \in \Gamma(U, \mathcal{E})$ we have

$$\tilde{f}_U(s)|_V = \tilde{f}_V\,(s|_V)$$

In (2.32) through (2.34), we introduce language to describe this situation.

Definition 2.32. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles over M. Then a *sheaf map* θ from $\mathcal{E}$ to $\mathcal{F}$ is a collection of functions

$$\{\theta_U : \Gamma(U, \mathcal{E}) \to \Gamma(U, \mathcal{F}) \quad | \quad U \subset M \quad \text{open}\}$$

such that:

i) For $\phi \in \mathcal{C}(M)$ and $s, t \in \Gamma(U, \mathcal{E})$, we have

$$\theta_U(\phi s + t) = \phi \theta_U(s) + \theta_U(t)$$

ii) For $V \subset U$ open and $s \in \Gamma(U, \mathcal{E})$, we have

$$\tilde{f}_U(s)|_V = \tilde{f}_V\,(s|_V)$$

We will use the notation

$$\theta : \tilde{\mathcal{E}} \to \tilde{\mathcal{F}}$$

to mean "θ is a sheaf map from $\mathcal{E}$ to $\mathcal{F}$".

Notation 2.33. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles over M. We write

$$\mathrm{Hom}_{VB}(\mathcal{E}, \mathcal{F})$$

for the set of all vector bundle maps from $\mathcal{E}$ to $\mathcal{F}$ and we write

$$\mathrm{Hom}_{Sh}(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$$

for the set of all sheaf maps from $\mathcal{E}$ to $\mathcal{F}$.

Definition 2.34. Given $f \in \mathrm{Hom}_{VB}(\mathcal{E}, \mathcal{F})$, we define the *associated sheaf map* $\tilde{f} \in \mathrm{Hom}_{Sh}(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ by

$$\tilde{f}_U(s)(m) = f(s(m))$$

Theorem 2.35. For each $\theta \in \mathrm{Hom}_{Sh}(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$, there exists a unique $f \in \mathrm{Hom}_{VB}(\mathcal{E}, \mathcal{F})$ such that $\theta = \tilde{f}$.

Proof. First, existence: Given $e \in \mathcal{E}_m$, we must give a formula for $f(e) \in \mathcal{F}_m$.

Choose an open set $U \subset M$ containing m and such that $\mathcal{E}|_U$ is trivial. Choose a section $s \in \Gamma(U, \mathcal{E})$ such that $s(m) = e$. Set

$$f(e) = \theta_U(s)(m) \tag{2.35.1}$$

If f is well-defined, it is easily seen to be a map of vector bundles and to satisfy $\theta = \tilde{f}$. So we need only check that f really *is* well- defined; that is, we need to check that the right side of (2.35.1) does not depend on the choices of U and s.

So let $V \subset M$ be another open set containing m such that $\mathcal{E}|_V$ is trivial and let $t \in \Gamma(V, \mathcal{E})$ be such that $t(m) = e$. (This is possible by (2.26.1). We need to show that

$$\theta_U(s)(m) = \theta_V(t)(m) \tag{2.35.2}$$

Clearly we can replace both U and V with $U \cap V$ and replace both s and t with their restrictions to $U \cap V$ without changing either side of equation (2.35.2). Thus we can assume that $U = V$.

Let

$$\{s_1, \ldots, s_k\}$$

be a global basis for $\mathcal{E}_U$, and use (2.30.1) to write

$$s = \sum_{i=1}^{k} a_i s_i \qquad\qquad t = \sum_{i=1}^{k} b_i s_i$$

where $a_i, b_i : U \to \mathbf{R}$. Then we have

$$\sum_{i=1}^{k} a_i(m) s_i(m) = s(m) = e = t(m) = \sum_{i=1}^{k} b_i(m) s_i(m) \tag{2.35.3}$$

Because the $s_i(m)$ form a basis for $\mathcal{E}_m$, we can conclude from (2.35.3) that

$$a_i(m) = b_i(m) \tag{2.35.4}$$

Now we have

$$\begin{aligned}
\theta_U(s)(m) &= \theta_U\left(\sum a_i s_i\right)(m) \\
&= \left(\sum a_i \theta_U(s_i)\right)(m) && \text{(by (2.31i) and (2.32.2))} \\
&= \sum a_i(m)\theta_U(s_i)(m) && \text{(by the definitions in (2.27)} \\
&= \sum b_i(m)\theta_U(s_i)(m) && \text{by (2.35.4)} \\
&= \left(\sum b_i \theta_U(s_i)\right)(m) && \text{by the definitions in (2.27)} \\
&= \theta_U\left(\sum b_i s_i\right)(m) && = \text{(by (2.31i) and (2.31ii))} \\
&= \theta_U(t)(m)
\end{aligned}$$

as needed.

This completes the proof of existence. For uniqueness, suppose that $\tilde{g} = \tilde{f}$. Then for any $e \in \mathcal{E}$, we can choose an open set U containing $m = p(e)$ such that $\mathcal{E}|_U$ is trivial. By (2.26.1), there exists $s \in \Gamma(U, \mathcal{E})$ with $s(m) = e$. Now the formula in (2.32) shows that $f(e) = g(e)$.

Remark 2.35.5 For $f, g \in \mathrm{Hom}_{VB}(\mathcal{E}, \mathcal{F})$ and $\phi \in \mathcal{C}(M)$, it is easy to check that we have

$$\widetilde{(f+g)} = \tilde{f} + \tilde{g}$$

$$\widetilde{\phi f} = \phi \tilde{f}$$

(All additions and multiplications in these formulas are defined via pointwise addition and multiplication in the vector spaces $\mathcal{F}_m$.) We summarize this situation by saying that the map $f \mapsto \tilde{f}$ is a *homomorphism over* $\mathcal{C}(M)$.

Unnecessary but Possibly Enlightening Remarks 2.36. We can describe the material of (2.32) through (2.35) in a slightly more abstract setting.

First, we make a definition: A *sheaf* on M consists of the following data:

i) For each open set $U \subset M$, a set $\mathcal{S}(U)$

ii) For each inclusion of open sets $V \subset U \subset M$, a function $\rho_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$

These data are required to satisfy:

i) For $W \subset V \subset U \subset M$ all open, we have $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$

ii) For every open set U and for every family of open sets $\{U_i\}$ whose union is U, the maps ρ_{U,U_i} induce a map

$$\mathcal{S}(U) \to \{\{s_i\} | s_i \in \mathcal{S}(U_i) \text{ and } \rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j) \text{ for all } i, j\}$$

We require this map to be one-one and onto.

(If we insist only on (i) and drop requirement (ii) we get the more general notion of a *presheaf*.)

For example, if $\mathcal{E}$ is a vector bundle, then we get the *associated sheaf* $\tilde{\mathcal{E}}$ by setting

$$\tilde{\mathcal{E}}(U) = \Gamma(U, \mathcal{E}) \qquad \text{and} \qquad \rho_{UV}(s) = s|_V$$

A sheaf $\mathcal{S}$ is called a *sheaf of modules* if:

i) For each U, $\mathcal{S}(U)$ is a module over $\mathcal{C}(M)$. This means we have a rule for adding two elements of $\mathcal{S}(U)$ and a rule for multiplying an element of $\mathcal{C}(U)$ times an element of $\mathcal{S}(U)$, and these rules satisfy the vector space axioms (I.1.1).

ii) The maps ρ_{UV} are module homomorphisms. This means that $\rho_{UV}(s + t) = \rho_{UV}(s) + \rho_{UV}(t)$ and $\rho_{UV}(\phi s) = \phi\rho_{UV}(s)$ for all $s, t \in \mathcal{S}(U)$, and $\phi \in \mathcal{C}(U)$.

For example, the sheaf associated to a vector bundle is a sheaf of modules.

If $\mathcal{S}$ and $\mathcal{T}$ are sheaves, a map

$$\theta : \mathcal{S} \to \mathcal{T} \tag{2.36.1}$$

is a collection of functions $\theta_U : \mathcal{S}(U) \to \mathcal{T}(U)$ satisfying

$$\rho^{\mathcal{S}}_{UV} \circ \theta_U = \theta_V(\mathcal{S}) \circ \rho^{\mathcal{T}}_{UV}$$

for all open sets $V \subset U \subset M$. (Here we have used superscripts $\mathcal{S}$ and $\mathcal{T}$ to distinguish the ρ maps that are part of the definition of $\mathcal{S}$ from those that are part of the definition of $\mathcal{T}$.) In case $\mathcal{S}$ and $\mathcal{T}$ are sheaves of modules, we also require the θ_U to be module homomorphisms.

Now let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles. In (2.31) we defined the notion of a *sheaf map* from $\mathcal{E}$ to $\mathcal{F}$ and denoted a typical sheaf map by the symbol

$$\theta : \tilde{\mathcal{E}} \to \tilde{\mathcal{F}} \tag{2.36.2}$$

even though the symbols $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ were themselves undefined. Now that we have defined the associated sheaves $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$, and now that we have defined maps of sheaves, the notation of (2.36.2) appears as a special case of (2.36.1). And of course, the two interpretations of (2.36.2)— the one given in (2.31) and the one given here—are equivalent.

3. From Functors to Vector Bundles

A functor F takes vector spaces to vector spaces and linear transformations to linear transformations. Our goal is to extend the domain of F so that it also takes vector bundles to vector bundles and maps of vector bundles to maps of vector bundles. It will turn out that in order to do this in a sensible way, we need to assume that F is *smooth* in a sense to be defined in 3.1. Fortunately, every functor in this book, and every functor you are likely to encounter in your lifetime, is smooth.

3A. Smooth Functors.

Definition 3.1. Let F be a functor and let V and W be vector spaces. Think of the rule $f \mapsto F(f)$ as a function

$$\begin{array}{ll} F : \mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W)) & \text{if } F \text{ is covariant} \\ F : \mathrm{Hom}(V, W) \to \mathrm{Hom}(F(W), F(V)) & \text{if } F \text{ is contravariant} \end{array} \tag{3.1.1}$$

We say that F is *smooth* if for every V and W, (3.1.1) is a smooth map of manifolds.

Exercise 3.1.2. Show that all the covariant and contravariant functors introduced in (I.3A) and (I.3B) are smooth.

Definition 3.2. Let F be a multifunctor. Consider the various ordinary functors that can be constructed by fixing all but one of the indexes as in (I.3.5.1). Then F is *smooth* if all of these ordinary functors are smooth.

Exercise 3.2.1. Show that all of the multifunctors of (I.3C) are smooth.

Lemma 3.3. Let $\mathcal{E}$ be a vector bundle over a manifold M. Then there exists a family of open sets $U_\alpha \subset M$ such that

a) $M = \bigcup U_\alpha$

b) All of the U_α are coordinate patches for M

c) All of the $\mathcal{E}|_{U_\alpha}$ are trivial.

Proof. From the definition of a manifold, $M = \bigcup V_\beta$ where the V_β are coordinate patches.

From the definition of a vector bundle, $M = \bigcup W_\gamma$ where all of the $\mathcal{E}|_{W_\gamma}$ are trivial.

Now let $\{U_\alpha\} = \{V_\beta \cap W_\gamma\}$. To see that the U_α are coordinate patches use (1.5), and to check the triviality condition use (2.7.4).

Exercise 3.3.1. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be vector bundles over M. Show that (3.3) remains true when c) is replaced by

c$'$) All of the $\mathcal{E}_i|_{U_\alpha}$ are trivial.

Definition 3.4. Let F be a smooth covariant functor (3.2) and $\mathcal{E}$ a vector bundle. We will define a new vector bundle $F(\mathcal{E})$.

First choose a family of sets $\{U_\alpha\}$ as in (3.3). Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M and let $\{(U_\alpha, \sigma_\alpha\})$ be a local trivialization for $\mathcal{E}$; thus we have

$$\sigma_\alpha : \mathcal{E}|_{U_\alpha} \to U_\alpha \times \mathbf{R}^k \tag{3.4.1}$$

For each α, (3.4.1) restricts to a linear transformation

$$\sigma_{\alpha m} : \mathcal{E}_m \to \mathbf{R}^k \tag{3.4.2}$$

where we have identified $\{m\} \times \mathbf{R}^k$ with $\mathbf{R}^k$ via projection on the second factor. Choose once and for all (and quite arbitrarily) an isomorphism

$$\theta : F(\mathbf{R}^k) \to \mathbf{R}^d$$

(one exists for some d by (I.1.10)) and define $\tau_{\alpha m}$ as the composition

$$\tau_{\alpha m} : F(\mathcal{E}_m) \xrightarrow{F(\sigma_{\alpha m})} F(\mathbf{R}^k) \xrightarrow{\theta} \mathbf{R}^d \tag{3.4.3}$$

Now set

$$F(\mathcal{E}) = \{(m, x) | m \in M \text{ and } x \in F(\mathcal{E}_m)\}$$

Let $q : F(\mathcal{E}) \to M$ to be projection on the first factor and set

$$F(\mathcal{E})_{U_\alpha} = q^{-1}(U_\alpha) \subset F(\mathcal{E})$$

Finally, define

$$\begin{array}{rccc} \tau_\alpha : & F(\mathcal{E})_{U_\alpha} & \to & U_\alpha \times \mathbf{R}^d \\ & (m,x) & \mapsto & (m, \tau_{\alpha m}(x)) \end{array}$$

We would like to say that the τ_α give $F(\mathcal{E})$ the structure of a vector bundle; the idea is that the τ_α constitute a local trivialization and the compositions

$$\begin{array}{ccccccc} F(\mathcal{E})_{U_\alpha} & \xrightarrow{\tau_\alpha} & U_\alpha \times \mathbf{R}^d & \to & \mathbf{R}^n \times \mathbf{R}^d & \approx & \mathbf{R}^{n+d} \\ & & (m,x) & \mapsto & (\phi_\alpha(m), x) & & \end{array}$$

constitute an atlas (so that $F(\mathcal{E}$ is a manifold).

Proposition (2.20.1) tells us exactly what it takes to make this work; the equivalence of (2.20.1(i)) and (2.20.1(iv)) says that our system of charts is a legitimate atlas if and only if the maps

$$\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Hom}(\mathbf{R}^d, \mathbf{R}^d)$$

given by

$$\lambda_{\alpha\beta}(m)(x) = \tau_{\underline{e}tam}\tau_{\alpha m}^{-1}(x)$$

are all smooth.

But $\lambda_{\alpha\beta}$ can be written as a composition

$$\begin{array}{ccccccc} U_{\alpha\beta} & \xrightarrow{\kappa_{\alpha\beta}} & \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^k) & \xrightarrow{F} & \mathrm{Hom}(F(\mathbf{R}^k), F(\mathbf{R}^k)) & \to & \mathrm{Hom}(\mathbf{R}^d, \mathbf{R}^d) \\ & & & & f & \mapsto & \theta \circ f \circ \theta^{-1} \end{array} \tag{3.4.4}$$

where the first map $\kappa_{\alpha_\beta}(m)(x) = \sigma_{\beta m}\sigma_{\alpha_m}^{-1}(x)$. By (1.6.2) it is now enough to show that each of the three maps in (3.4.4) is smooth.

The equivalence of (2.20.1(i)) and (2.20.1(iv)) shows that $\kappa_{\alpha\beta}$ is smooth, because $\mathcal{E}$ is known to be a vector bundle. The map F is smooth because F is assumed to be a smooth functor. The map $f \mapsto \theta \circ f \circ \theta^{-1}$ is linear and therefore smooth by (1.6.4).

Definition 3.5. Let $f : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles, and let F be a smooth covariant functor. We define a map of vector bundles

$$F(f) : F(\mathcal{E}) \to F(\mathcal{F})$$

by

$$F(f)(m,x) = (m, F(f)(x)) \tag{3.5.1}$$

You should of course verify that (3.5.1) is a map of vector bundles.

Definition 3.6. Let F be a smooth contravariant functor (3.2) and let $\mathcal{E}$ be a vector bundle. We define a new vector bundle $F(\mathcal{E})$ exactly as in (3.4), with just one change: Replace (3.4.3) by

$$\tau_{\alpha m} : F(\mathcal{E}_m) \xrightarrow{F(\sigma_{\alpha m})^{-1}} F(\mathbf{R}^k) \xrightarrow{\theta} \mathbf{R}^d \tag{3.6.1}$$

Definition 3.7. Let $f : \mathcal{E} \to \mathcal{F}$ be a map of manifolds and let F be a smooth contravariant functor. We define a map of vector bundles

$$F(f) : F(\mathcal{F}) \to F(\mathcal{E})$$

by equation (3.5.1).

Definition 3.8 Let F be a smooth multifunctor (3.2) of type (p,q) where $p+q=n$. Given vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$, define

$$F(\mathcal{E}_1, \ldots, \mathcal{E}_n) = \{(m,x) | m \in M \text{ and } x \in F(\mathcal{E}_1, \ldots, \mathcal{E}_n)\}$$

To give $F(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ the structure of a vector bundle, mimic the construction of (3.4) as follows:

Use (3.3.1) to choose an atlas for M. Let $\sigma_{\alpha i} : \mathcal{E}_i \to U_\alpha \times \mathbf{R}^{k_i}$ be a trivialization. Let $\sigma_{\alpha i m} : \mathcal{E}_{im} \to \mathbf{R}^{d_i}$ be the restriction of $\sigma_{\alpha i}$ to the fiber. Choose once and for all an isomorphism $\theta : F(\mathbf{R}^{k_1}, \ldots \mathbf{R}^{k_n}) \to \mathbf{R}^d$. Define $\tau_{\alpha m} = \theta \circ F(\sigma_{\alpha 1 m}, \ldots, \sigma_{\alpha p m}, \sigma^{-1}_{\alpha(p+1)m}, \ldots, \sigma^{-1}_{\alpha n m})$. Map $\tau_\alpha : F(\mathcal{E}_1, \ldots, \mathcal{E}_n) \to U_\alpha \times \mathbf{R}^d$ by $(m,x) \mapsto (\phi_\alpha(m), \tau_{\alpha m}(x))$, and check that the τ_α give $F(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ the structure of a vector bundle.

Definition 3.9. Let F be a smooth multifunctor of type (p,q) where $p+q=n$ and suppose we are given vector bundle maps $f_i : \mathcal{E}_i \to \mathcal{F}_i$ $(i = 1, \ldots p)$ and $f_i : \mathcal{F}_i \to \mathcal{E}_i$ $(i = p+1, \ldots, n)$. Define a vector bundle map

$$F(f_1, \ldots f_n) : F(\mathcal{E}_1, \ldots, \mathcal{E}_n) \to F(\mathcal{F}_1, \ldots, \mathcal{F}_n)$$

by equation (3.5.1).

Examples 3.10. Given vector bundles $\mathcal{E}$ and $\mathcal{F}$, the Hom and Tensor functors (I.3.2) give rise to bundles $Hom(\mathcal{E}, \mathcal{F})$ and $\mathcal{E} \otimes \mathcal{F}$.

Applying the multifunctor $\mathcal{T}^{p,q}$ of (I.5.1) with $\mathcal{E}$ in every slot, we get the (p, q)-*tensor bundle of* $\mathcal{E}$ denoted $T^{p,q}(\mathcal{E})$.

The dual functor (I.3.4.3) gives rise to a bundle $\mathcal{E}^*$.

Proposition 3.11. There is an isomorphism of vector bundles

$$\mathcal{E}^* \to Hom(\mathcal{E}, M \times \mathbf{R})$$

Proof. By construction

$$(\mathcal{E}^*)_m = (\mathcal{E}_m)^* = \mathrm{Hom}(\mathcal{E}_m, \mathbf{R})$$

whereas

$$Hom(\mathcal{E}, M \times \mathbf{R})_m = \mathrm{Hom}(\mathcal{E}_m, (M \times \mathbf{R})_m) = \mathrm{Hom}(\mathcal{E}_m, \{m\} \times \mathbf{R})$$

Thus we can map

$$\begin{array}{ccc} (\mathcal{E}^*)_m & \to & Hom(\mathcal{E}, M \times \mathbf{R})_m \\ f_m & \mapsto & \big(e_m \mapsto (m, f_m(e_m))\big) \end{array}$$

Proposition 3.12. Let $F_1, \ldots, F_n$ be functors (each either covariant or contravariant), let G be a multifunctor of type (p, q) where $p + q = n$, and let $G \circ F$ be the composed functor (I.3.8). Then for any vector bundles $\mathcal{E}_1, \mathcal{E}_n$, we have

$$(G \circ F)(\mathcal{E}_1, \ldots, \mathcal{E}_n) = G(F(\mathcal{E}_1, \ldots, \mathcal{E}_n))$$

Remark 3.12.1. Proposition (3.12) makes it possible for us to assign an unambiguous meaning to, say,

$$\mathcal{E} \otimes \mathcal{E}^* \tag{3.12.1.1}$$

Let F_1 be the identity functor, let F_2 be the dual functor, and let G be the tensor product bifunctor. Then it's not immediately obvious whether (3.12.1.1) refers to the bundle

$G(F_1(\mathcal{E}), F_2(\mathcal{E}))$ or to the bundle $(G \circ F)(\mathcal{E}, \mathcal{E})$. But according to Proposition (3.12), the distinction makes no difference.

3B. Hom, Sections and Duality

Recalled Notation 3.13. Recall from (2.26) that $\Gamma(M, Hom(\mathcal{E}, \mathcal{F}))$ is the set of sections of $Hom(\mathcal{E}, \mathcal{F})$ and recall from (2.32) that $\mathrm{Hom}_{VB}(\mathcal{E}, \mathcal{F}))$ is the set of vector bundle maps from $\mathcal{E}$ to $\mathcal{F}$.

Proposition 3.14. There is a one-one correspondence

$$\begin{array}{ccc} \Gamma(M, Hom(\mathcal{E}, \mathcal{F})) & \leftrightarrow & \mathrm{Hom}_{VB}(\mathcal{E}, \mathcal{F})) \\ \xi & \mapsto & \Big(e_m \mapsto \xi(m)(e_m) \in \mathcal{F}_m\Big) \\ \big(m \mapsto f_m\big) & \leftarrow\!\shortmid & f \end{array}$$

(Here ξ and f are arbitrary elements of the left and right-hand sides, f_m is as in (2.6.1), and e_m is an arbitrary element of $\mathcal{E}_m \subset \mathcal{E}$.)

We need to check that each direction of this correspondence takes smooth maps to smooth maps. For this it is sufficient to restrict to coordinate patches on which $\mathcal{E}$ and $\mathcal{F}$ are trivial bundles, and then the desired result reduces easily to (2.19.1).

Corollary 3.15. There is a one- one correspondence

$$\Omega : \Gamma(M, Hom(\mathcal{E}, \mathcal{F})) \to \mathrm{Hom}_{Sh}(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \tag{3.15.1}$$

which can be described as follows: For $\xi \in \Gamma(M, Hom(\mathcal{E}, \mathcal{F}))$, and for $U \subset M$ open, we define

$$\begin{array}{cccc} \Omega(\xi)_U : & \Gamma(U, \mathcal{E}) & \to & \Gamma(U, \mathcal{F}) \\ & s & \mapsto & \Big(m \mapsto \xi(m)(s(m))\Big) \end{array}$$

Proof. Combine (2.34) and (3.14).

Corollary 3.16. There is a one- one correspondence

$$\Omega : \Gamma(M, \mathcal{E}^*) \to \mathrm{Hom}_{Sh}(\tilde{\mathcal{E}}, \widetilde{M \times \mathbf{R}})$$

which can be described as follows: For $\xi \in \Gamma(M, \mathcal{E}^*)$ and for $U \subset M$ open, we have

$$\begin{array}{rccc} \Omega(\xi)_U: & \Gamma(U, \mathcal{E}) & \to & \Gamma(U, \widetilde{M \times \mathbf{R}}) \\ & s & \mapsto & \Big(m \mapsto (m, \xi(m)(s(m)))\Big) \end{array} \tag{3.16.1}$$

Proof. Specialize (3.15) to $\mathcal{F} = M \times \mathbf{R}$ and combine with (3.11).

Remark 3.16.2. An element s of $\Gamma(U, \widetilde{M \times \mathbf{R}})$ is of the form $m \overset{s}{\longmapsto} (m, \phi(m))$ for some smooth map $\phi : U \to \mathbf{R}$. Thus (3.16) is really a proposition about smooth real-valued maps. The remainder of Section (3C) makes this reinterpretation precise.

Notation 3.17. Let $\mathcal{E}$ be a vector bundle over M. Let

$$\xi \in \Gamma(M, \mathcal{E}^*) \qquad \text{and} \qquad s \in \Gamma(U, \mathcal{E})$$

Define a smooth map

$$< \xi, s >: M \to \mathbf{R}$$

by the formula

$$< \xi, s > (m) = \xi(m)(s(m))$$

Note that $< \xi, s >$ is smooth because it is the composition of the right side of (3.16.1) with projection on the second factor.

Notation 3.18. For $\xi \in \Gamma(M, \mathcal{E}^*)$, and for $U \subset M$ open, define a function

$$\begin{array}{rccc} < \xi, - >: & \Gamma(U, \mathcal{E}) & \to & \mathcal{C}(U) \\ & s & \mapsto & < \xi, s > \end{array} \tag{3.18.1}$$

Proposition 3.19. For any section $\xi \in \Gamma(M, \mathcal{E}^*)$, the map $< \xi, - >$ satisfies the following properties:

i) For $s, t \in \Gamma(U, \mathcal{E}^*)$ and $\phi \in \mathcal{C}(M)$,

$$< \xi, \phi s + t >= \phi < \xi, s > + < \xi, t >$$

ii) For $V \subset U$ and $s \in \Gamma(U, \mathcal{E})$, let $s|_V$ be the restriction of s to V so that $s|_V \in \Gamma(V, \mathcal{E})$ Then

$$< \xi, s > |_V =< \xi, s|_V >\in \mathcal{C}(V)$$

We will next show that any function satisfying these properties is of the form $<\xi, ->$ for some $\xi \in \Gamma(M, \mathcal{E}^*)$.

Theorem 3.20. Suppose we are given, for each open set $U \subset M$, a function

$$\zeta_U : \Gamma(U, \mathcal{E}) \to \mathcal{C}(M)$$

such that

i) For $s, t \in \Gamma(U, \mathcal{E})$ and $\phi \in \mathcal{C}(M)$,

$$\zeta_U(\phi s + t) = \phi \zeta_U(s) + \zeta_U(t)$$

ii) For $V \subset U$ and $s \in \Gamma(U, \mathcal{E})$,

$$\zeta_U(s)|_V = \zeta_V\left(s|_V\right) \in \mathcal{C}(V)$$

Then there exists a unique section $\xi \in \Gamma(M, \mathcal{E}^*)$ such that for all U and for all $s \in \Gamma(U, \mathcal{E})$,

$$\zeta_U(s) =< \xi, s >$$

Proof. For each open U, define

$$\theta_U : \Gamma(U, \mathcal{E}) \to \Gamma(U, M \times \mathbf{R})$$

by

$$\theta_U(s)(m) = (m, \zeta_U(s)(m)) \tag{3.20.1}$$

The maps θ_U constitute a sheaf map (2.32)

$$\theta : \widetilde{\mathcal{E}} \to (\widetilde{M \times \mathbf{R}})$$

According to (3.16), there is a unique section $\xi : M \to \mathcal{E}^*$ such that for all U and for all m,

$$\theta_U(m) = (m, \xi(m)(s(m))) = (m, < \xi, s > (m)) \tag{3.20.2}$$

Now combine (3.20.1) with (3.20.2) and project on the second factor to get the theorem.

Corollary 3.21. Let $\{\zeta_U\}$ be any collection of maps satisfying (3.20i) and (3.20.ii). Let U be an open set, let $s, t \in \Gamma(\mathcal{E}, U)$ be sections, and let m be a point in U. Then

$$s(m) = t(m) \qquad \text{implies} \qquad \zeta_U(s)(m) = \zeta_U(t)(m)$$

3C. Tensor Product Bundles

The results of this section are similar in flavor to those of Section 3B.

Definition 3.22. Given sections $s_1, \ldots s_n \in \Gamma(M, \mathcal{E})$, define a section

$$\begin{array}{rccl} s_1 \otimes \cdots \otimes s_n : & M & \to & \mathcal{E} \otimes \cdots \otimes \mathcal{E} \\ & m & \mapsto & s_1(m) \otimes \cdots \otimes s_n(m) \quad \in \mathcal{E}_m \otimes \cdots \otimes \mathcal{E}_m \end{array}$$

Proposition 3.23 If $\{s_1, \ldots, s_k\}$ is a global basis (2.29) for $\mathcal{E}$ then every section

$$s \in \Gamma(\mathcal{E} \otimes \ldots \otimes \mathcal{E})$$

is of the form

$$s = \sum_{i_1, \ldots, i_k} \phi_{i_1, \ldots, i_k} s_1 \otimes \cdots \otimes s_k \tag{3.23.1}$$

for uniquesmooth functions $\phi_{i_1, \ldots, i_k} : M \to \mathbf{R}$.

Proof. For any m, (I.2.13) allows us to write

$$s(m) = \sum_{i_1, \ldots, i_k} \phi_{i_1, \ldots, i_k}(m)(s_i \otimes \ldots \otimes s_k)(m) \tag{3.23.2}$$

for some unique $\phi_{i_1, \ldots, i_k}(m) \in \mathbf{R}$. Use charts to check that smoothness of the $\phi_{i_1, \ldots, i_k}$ is equivalent to the smoothness of s.

Proposition 3.24. Suppose that for every open $U \subset M$, we are given a map

$$R_U : \Gamma(U, \mathcal{E}) \times \cdots \times \Gamma(U, \mathcal{E}) \to \Gamma(U, \mathcal{E})$$

satisfying the following properties:

i) For $\phi : M \to \mathbf{R}$ smooth, and for any i, we have

$$R_U(s_1, \dots, \phi s_i + t_i, \dots, s_k) = \phi R_U(s_1, \dots, s_i, \dots, s_k) + R_U(s_1, \dots, t_i, \dots, s_k)$$

ii) For $V \subset U$ open, we have

$$R_U(s_, \dots, s_k)|_V = R_V(s_1|_V, \dots, s_k|_V)$$

Then there is a unique sheaf map (2.32)

$$\widetilde{R} : \overbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}} \to \widetilde{\mathcal{E}}$$

such that for all open sets U and all sections $s_1, \dots, s_k \in \Gamma(U, \mathcal{E})$ we have

$$R_U(s_1, \dots, s_k) = \widetilde{R}_U(s_1 \otimes \cdots \otimes s_k)$$

Proof. Given a section

$$s \in \Gamma(U, \mathcal{E} \otimes \cdots \otimes \mathcal{E})$$

we must define $R_U(s)$.

If $\mathcal{E}|U$ has a global basis $\{s_1, \dots, s_k\}$ write s in the form (3.23.1) and define

$$R_U(s) = \sum_{i_1, \dots, i_k} \phi_{i_1, \dots, i_k} R(s_1, \dots, s_l) \tag{3.24.1}$$

Otherwise, cover U with open subsets V_i such that $\mathcal{E}|_{V_i}$ is trivial, and use (3.24.1) to define maps R_{V_i}. For any $m \in M$, choose i with $m \in V_i$ and define

$$R_U(s)(m) = R_{V_i}(s|V_i)(m)$$

This is independent of i and hence well- defined. Moreover, $R_U(s) : M \to \mathcal{E}$ is smooth because its restriction to each V_i is smooth.

Corollary 3.25. Given maps satisfying (3.24i) and (3.24ii), there is a unique vector bundle map

$$R : \mathcal{E} \otimes \cdots \otimes \mathcal{E} \to \mathcal{E}$$

such that for all U, all $s_1, \ldots, s_k$ and all m,

$$R_U(s_1, \ldots, s_k)(m) = R(s_1 \otimes \cdots \otimes s_k)(m)$$

Proof. Use (3.24) and (2.35).

Corollary 3.26. Given maps satisfying (3.24i) and (3.24ii), given a point $m \in$M, and given sections

$$s_1, \ldots, s_k, t_1, \ldots, t_k \in \Gamma(M, \mathcal{E})$$

such that

$$s_i(m) = t_i(m) \in T_m\mathcal{E} \qquad \text{for all } i$$

we can conclude that

$$R_U(s_1, \ldots, s_k) = R_U(t_1, \ldots, t_k)$$

for all open U containing m.

Corollary 3.27. Given maps satisfying (3.24i) and (3.24ii), there is a section

$$R \in \Gamma(M, Hom(\mathcal{E} \otimes \cdots \otimes \mathcal{E}, \mathcal{E}))$$

such that

$$R_U(s_1, \ldots, s_k)(m) = R(m)(s_1(m) \leq times \cdots \otimes s_k(m)) \in \mathcal{E}_m$$

for all $m \in M$, all open U containing m and all sections $s_1, \ldots, s_k$.

Proof. Use (3.24) and (3.14).

Remark 3.27.1 We have used the same symbol R for the vector bundle map of (3.26) and the section of (3.27).

Remark 3.28. All the results of Section 3B generalize easily to the case where $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are (possibly distinct) vector bundles and s_i is a sectionof $\mathcal{E}_\rangle$.

3D. Naturality

Definition 3.29. Let F and G be (covariant or contravariant) functors and let

$$\phi : F \Rightarrow G \tag{3.29.1}$$

be a natural transformation (I.4.2). Then ϕ induces a map of vector bundles

$$\begin{array}{rccc} \phi : & F(\mathcal{E}) & \rightarrow & G(\mathcal{E}) \\ & (m, x) & \mapsto & \phi_{E_m}(x) \end{array} \tag{3.29.2}$$

Similarly, if (3.29.1) is a natural transformation of multifunctors, we get

$$\phi : F(\mathcal{E}_1, \ldots, \mathcal{E}_n) \rightarrow G(\mathcal{E}_1, \ldots, \mathcal{E}_n) \tag{3.22.3}$$

by using equation (3.29.2).

Exercise 3.29.4 Show that if ϕ is a natural isomorphism (I.4.3) then the induced map (3.29.1) (or, in the case of multifunctors, (3.29.3)) is an isomorphism of vector bundles.

Remark 3.30. All the natural isomorphisms of (I.4.12) and all their consequences (see for example I.4.13 and I.5.2.3) carry over to isomorphisms of vector bundles. Thus, for example, if $\mathcal{E}$ $\mathcal{F}$ and $\mathcal{G}$ are vector bundles, we have isomorphisms $\mathcal{E} \approx \mathcal{E}^{**}$, $\mathcal{E} \otimes \mathcal{F} \approx \mathcal{F} \otimes \mathcal{E}$, $Hom(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \approx Hom(\mathcal{E}, Hom(\mathcal{F}, \mathcal{G}))$, etc.

Remark 3.31. Any vector space V is isomorphic to its dual V^* by (I.2.6). However, a vector bundle $\mathcal{E}$ is in general *not* isomorphic to its dual $\mathcal{E}^*$. This reflects the fact that there is no *natural* isomorphism $V \rightarrow V^*$ so the construction of (3.22) does not apply.

4. Cotangent Spaces, Tangent Spaces and Tensor Spaces

4A. Cotangents.

Definition 4.1. Let M be a manifold and let $m \in M$ be a point. A *real function defined near* m is a pair (U, f) where U is an open set containing m and $f : U \rightarrow \mathbf{R}$ is a map.

Example 4.1.1. Let (U, ϕ) be an n-dimensional chart on M and $f : \mathbf{R}^n \rightarrow \mathbf{R}$ an infinitely differentiable map. Then $(U, f \circ \phi)$ is a real function defined near any element of U.

Vector Space Structure 4.2. The real functions defined near m form a vector space under the following operations:

$$(U, f) + (V, g) = (U \cap V, f|_{U \cap V} + g|_{U \cap V})$$

$$\alpha(U, f) = (U, \alpha f)$$

Definition 4.3. Let M be a manifold and m a point in M. Let $\mathcal{V}_m$ be the space of functions defined near m. We want to define a subspace $\mathcal{W}_m$ consisting of those functions that "become zero near m". More precisely, (U, f) is in $\mathcal{W}_m$ if there exists an open set W with $m \in W \subset U$ such that $f|_W$ is identically zero.

The space of *germs of functions at* m is the space $\mathcal{G}_m = \mathcal{V}_m/\mathcal{W}_m$.

Any germ $\xi \in \mathcal{G}_m$ can be represented by a pair (U, f) as in (4.1) and we define $\xi(m) = f(m)$. (Check that this is well-defined! In other words, check that if ξ is also represented by (V, g) then $f(m) = g(m)$.)

4.3.1. In addition to the vector space operations on germs, there is also a way to multiply one germ by another. Represent germs ξ_1 and ξ_2 by pairs (U_1, f_1) and (U_2, f_2) and define $\xi_1\xi_2$ to be the germ represented by

$$(U_1 \cap U_2, f_1|_{U_1 \cap U_2} f_2|_{U_1 \cap U_2})$$

Proposition 4.4. Let U be an open set containing m. Then any germ at m can be represented by a pair (V, f) with $V \subset U$.

Definition 4.5. Let M be a manifold, $m \in M$ a point, and $\mathcal{G}_m$ the space of germs at m. Let $\mathcal{M}_m \subset \mathcal{G}_m$ be the subspace of those germs ξ such that $\xi(m) = 0$. Let $\mathcal{M}_m^2 \subset \mathcal{M}_m$ be the space of all germs of the form

$$\sum_{i=1}^{r} \xi_i \rho_i$$

where the ξ_i and ρ_i are all in $\mathcal{M}_m$. You can think of $\mathcal{M}_m^2$ as the space of those germs that "vanish to second order" at m.

Now set $T_m^*M = \mathcal{M}_m/\mathcal{M}_m^2$ and call T_m^*M the *cotangent space* to M at m.

Exercise 4.5.1. Suppose $m \in U \subset M$ with U open in M. Show that there is an isomorphism of vector spaces $T_m^*U \approx T_m^*M$.

Notation 4.5.2. Suppose (U, f) is a function defined near m. Then $(U, f - f(m))$ represents a germ in $\mathcal{M}_m$ and hence an element of the cotangent space T_m^*M. We denote this element by the symbol df_m. When the point m is unambiguously fixed in the discussion, we will sometimes abbreviate df_m as just df (but later on, in (6.2), the symbol df will mean something slightly different!).

Propostion 4.6. Let f and g be functions near $m \in M$.

i) If f and g differ by a constant, then $df = dg$. In particular, if α is a constant function, then $d\alpha = 0$.

ii) If α is a constant then $d(\alpha f) = \alpha df$

iii) $d(f + g) = df + dg$

iv) $d(fg) = f(m)dg + g(m)df$

Proof. (i), (ii) and (iii) follow immediately from the definition.

For (iv), write

$$fg = (f - f(m))(g - g(m)) + f(m)g + g(m)f + f(m)g(m)$$

and note that the first term on the right is in $\mathcal{M}_m^2$ so that

$$d(fg) = d\left(f(m)g + g(m)f + f(m)g(m)\right)$$

which can be evaluated by (ii) and (iii).

Definition 4.7. Let $\phi : M \to N$ be a map of manifolds. For each $m \in M$, we define a linear transformation

$$\phi_m^* : T_{\phi(m)}^*N \to T_m^*M$$

by

$$df \mapsto d(f \circ \phi)$$

(Here f is a function defined on some open set U containing $n \in N$ and $f \circ \phi$ is defined on $\phi^{-1}(U) \subset M$.)

Proposition 4.8. Let

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

be maps of manifolds. Then for each $m \in M$

$$(\psi \circ \phi)^*_m = \psi^*_{\phi(m)} \circ \phi^*_m$$

Corollary 4.8.1. If ϕ is a diffeomorphism, then ϕ^*_m is an isomorphism of vector spaces.

Proof. Apply 4.8 to the case $\psi = \phi^{-1}$.

Exercise 4.8.2. Use (4.7) and (4.8) to define a contravariant functor that takes each vector space V to the cotangent space T_0^*V.

Remarks 4.8.2.1. We can generalize the result of (4.8.2) if we extend the definition of "functor" as follows: Define a *pointed vector space* to be a pair (V, m) where V is a vector space and m is an element of V. Define a *pointed linear transformation* $\phi : (V, m) \to (W, n)$ to be a linear transformation $\phi : V \to W$ such that $\phi(m) = n$. Define covariant and contravariant *functors of pointed vector spaces* just as we defined functors in (I.3.1) and (I.3.3), except that the "inputs", instead of being vector spaces and linear transformations, are *pointed* vector spaces and *pointed* linear transformations. Then (4.7) and (4.8) allow us to define a contravariant functor that takes the pointed vector space (V, m) to the pair $(T_m^*V, 0)$.

Proposition 4.9. The cotangent space $T_0^*\mathbf{R}^n$ is n- dimensional.

Proof. Let x_i be the i^{th} coordinate function on $\mathbf{R}^n$. The proposition follows immediately from:

Claim 4.9.1. The cotangents dx_i form a basis for the vector space $T_0^*\mathbf{R}^n$.

Per (I.1.5.2), the claim encompasses two subclaims.

Subclaim 4.9.1.1. The dx_i span $T_0^*\mathbf{R}^n$.

Proof. An arbitrary cotangent vector can be represented by a function $f : V \to \mathbf{R}$ with V open. We can replace V by any smaller open set containing m, so in particular we may assume V is an open disk.

We must show that df is a linear combination (I.1.5.2) of the dx_i.

For this, note that

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{\partial}{\partial t} f(tx)dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx)x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^n x_i g_i(x) \end{aligned} \tag{4.9.1.1.1}$$

where

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \tag{4.9.1.1.2}$$

From (4.6) and (4.9.1.1.1), we get

$$df = \sum_{i=1}^n g_i(0)dx_i + \sum_{i=1}^n x_i(0)dg_i \tag{4.9.1.1.3}$$

where the rightmost term vanishes because $x_i(0) = 0$.

This establishes the claim.

Remark 4.9.1.1.4. It follows from (4.9.1.1.2) that

$$g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

Together with (4.9.1.1.3), this gives the useful formula

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0)dx_i \tag{4.9.1.1.5}$$

Subclaim 4.9.2. The dx_i are linearly independent.

Proof. Suppose

$$\sum_{i=1}^{n} \alpha_i dx_i = 0$$

By (4.6), or (4.9.1.1.5), the left hand side is equal to

$$d\left(\sum_{i=1}^{n} \alpha_i x_i\right)$$

so that (after restricting to a suitably small neighborhood of zero) we have

$$\sum_{i=1}^{n} \alpha_i x_i = \sum_{j=1}^{r} \xi_i \rho_i \tag{4.9.2.1}$$

where $\xi_i(0) = \rho_i(0) = 0$.

Differentiating each side of (4.9.2.1) with respect to x_i (and using the ordinary product rule from advanced calculus) gives

$$\alpha_i = 0$$

for all i, which is what is needed.

Exercise 4.9.3. Let V be a vector space. Show that the map

$$\begin{array}{rccc} \xi_V : & V^* & \to & T_0^* V \\ & f & \mapsto & df \end{array}$$

is an isomorphism. (Here f is a typical element of V^*, i.e. a linear map from V to $\mathbf{R}$.) Now show that the maps ξ_V constitute a natural transformation from the dual functor (I.3.4.3) to the functor you defined in (4.8.2).

(Hint: Start with the case $V = \mathbf{R}^n$ and use (4.9.1); now use the fact that any vector space is isomorphic to some $\mathbf{R}^n$.)

Remark 4.9.3.1. It follows from (4.9.3) that $T_0^* V$ is *naturally* isomorphic to V^*. In the future, we will use this natural isomorphism to identify $T_0^* V$ with V^*.

Remarks 4.9.3.2 Refer to (4.8.2.1) for the definition of a *functor of pointed vector spaces*. We can mimic the definition of natural transformation to define a *natural transformation between functors of pointed vector spaces* and generalize (4.9.3.1) to establish a natural isomorphism

$$\begin{array}{rccc} \xi_{V,m} : & V^* & \to & T_m^* V \\ & f & \mapsto & df_m \end{array}$$

for any pointed vector space (V, m).

Theorem 4.10. If m is a point in an n-dimensional manifold M, then the cotangent space $T_m^* M$ is n- dimensional.

Proof. Choose a coordinate patch U containing m and a chart $\phi : U \to \mathbf{R}^n$. Adding an appropriate constant to ϕ, we can assume that $\phi(m) = 0 \in \mathbf{R}^n$. Set $\Omega = \phi(U)$.

Then we have

$$
\begin{aligned}
T_m^* M &\approx T_m^* U && \text{by } (4.5.1) \\
&\approx T_0^* \Omega && \text{by } (4.8.1) \\
&\approx T_0^* \mathbf{R}^n && \text{by } (4.5.1)
\end{aligned}
$$

and this is n-dimensional by (4.9).

Remark 4.10.1. Let (U, ϕ) be any chart with $m \in U$, let x_i be the i^{th} coordinate function on $\mathbf{R}^n$, and let

$$x_i^\phi = x_i \circ \phi : U \to \mathbf{R}$$

Then $dx_i^\phi = \phi^*(dx_i)$, so that (4.9.1), together with the proof of (4.10), shows that the dx_i^ϕ form a basis for $T_m^* M$.

Warning 4.10.1.1. This notation suppresses the dependence of dx_i^ϕ on m. In cases where m is not clearly established by context, we will write $dx_i^\phi(m)$ instead of dx_i^ϕ.

4B. Tangents

Definition 4.11. Let M be a manifold and m a point in M. We define the *tangent space to M at m* $T_m M$ to be the dual (I.2.1.5) of the cotangent space:

$$T_m M = (T_m^* M)^*$$

Remark 4.11.1. If M is an n- dimensional manifold, then $T_m M$ is an n- dimensional vector space by (4.10) and (I.2.6).

Remark 4.11.2. The dual of the tangent space is the double dual of the cotangent space, which, in accordance with (I.4.4.1.1), can be identified with the cotangent space itself. Thus we write

$$(T_m M)^* = T_m^* M \tag{4.11.2.1}$$

Intuition 4.11.3. To envision the tangent space, start with the case $M = \mathbf{R}^n$. A tangent vector is a linear map that takes cotangents df to scalars. One example of such a map is

$$df \mapsto \frac{\partial f}{\partial x_i}(m)$$

where x_i is any one of the coordinate functions on $\mathbf{R}^n$.

There are n such maps, and they are linearly independent, so by (4.11.1) they span the tangent space $T_m \mathbf{R}^n$. Thus every tangent vector is of the form

$$df \mapsto \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}(m)$$

which is an example of a "directional derivative" operator. Thus every tangent vector is identified with a directional derivative at m, and hence with a "direction" at m.

If M is a manifold other than a vector space, you can still essentially retain intuition. For example, let M be the 2-sphere $\mathbf{S}^2 \subset \mathbf{R}^n$. Let m be a point, U an open set containing m, and $f : U \to \mathbf{R}$ a smooth function. Then the various tangent vectors at m represent directions in which f can be differentiated. You should visualize these directions as physical arrows tangent to the sphere at m.

Notation and Proposition 4.12. Let U be a coordinate patch containing m and $\phi : U \to \mathbf{R}^n$ a chart. Let $dx_i^\phi \in T_m^* M$ be as in (4.10.1). For fixed i, consider the element $\xi_i \in T_m M$ defined by

$$\xi_i(dx_j^\phi) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Then by 4.11.3 it makes sense to introduce the notation

$$\frac{\partial}{\partial x_i^\phi} = \xi_i$$

and by 4.10.1 together with (I.2.1.6.1) we see that

$$\left\{ \frac{\partial}{\partial x_1^\phi}, \ldots, \frac{\partial}{\partial x_n^\phi} \right\}$$

forms a basis for $T_m M$.

Remark 4.12.1. Although the cotangent vector dx_i^ϕ (4.10.1) can be constructed from knowledge of this coordinate function, the tangent vector $\partial / \partial x_i^\phi$ can not be. The tangent vector $\partial / \partial x_i^\phi$, depends on the coordinate functions x_j^ϕ for *all* values of j. (See I.2.6.1.3).

More Notation 4.13. For f a function defined near m, write

$$\frac{\partial f}{\partial x_i^\phi}(m) = \frac{\partial}{\partial x_i^\phi}(df)$$

(The right side of the equation shows a tangent vector in $T_m M$ acting on a cotangent vector in $T_m^* M$; the notation hides the dependence on m.)

Proposition 4.14. Let (U, ϕ) be a chart on M and let $f : U \to \mathbf{R}$ be a smooth map. Then for every $m \in M$ we have

$$\frac{\partial f}{\partial x_i^\phi}(m) = \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(\phi(m))$$

(Note that the right hand side is an ordinary partial derivative as defined in advanced calculus).

Proof. Because the dx_j^ϕ form a basis for $T_m^* M$, it is enough to consider the case $f = x_j^\phi = x_j \circ \phi$, so that

$$f \circ \phi^{-1} = x_j : \mathbf{R}^n \to \mathbf{R}$$

is just the j^{th} coordinate map. By definition, the left side of the equation is 1 or 0 depending on whether i does or does not equal j; by a trivial computation, the right side is the same.

Corollary 4.14.1. Let M be a manifold, (U, ϕ) a chart and $f : U \to \mathbf{R}$ a smooth map, Then for each i, the map

$$\begin{aligned} \frac{\partial f}{\partial x_i^\phi} : \quad & U \to \mathbf{R} \\ & u \mapsto \frac{\partial f}{\partial x_i^\phi}(m) \end{aligned}$$

is smooth.

Proof. The identity map is a chart on $\mathbf{R}$, so by (1.6.3) it suffices to show infinite differentiability of

$$\frac{\partial f}{\partial x_i^\phi} \circ \phi^{-1} : \mathbf{R}^n \to \mathbf{R} \tag{4.14.1}$$

By (4.14), (4.14.1.1) is just the partial derivative map

$$y \mapsto \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(y)$$

which is infinitely differentiable because $f \circ \phi^{-1}$ is.

Corollary 4.14.2. For a chart (U, ϕ) and a smooth map $f : U \to \mathbf{R}$, we have (at any point $m \in U$)

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i^\phi}(m) dx_i^\phi$$

Proof. First replace f by $f \circ \phi^{-1}$ in (4.9.1.1.5). Then apply the map ϕ^* (4.7) to both sides of the resulting equation and use (4.14).

Example 4.14.3. Let $M = \mathbf{S}^2$ be the two-sphere, and let (Ω, ϕ) be the chart described in (1.3.7.) That is, $\Omega \subset \mathbf{S}^2$ is the complement of the set $Z = \{(x, y, z) \in \mathbf{S}^2 | x \leq 0, y = 0\}$ and ϕ^{-1} is the map

$$\begin{array}{rccc} \phi^{-1} : & (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) & \to & \Omega \\ & (u, v) & \mapsto & (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)) \end{array}$$

The vectors $\partial/\partial u^\phi$ and $\partial/\partial v^\phi$ generate the tangent space $T_m\mathbf{S}^2$.

Let x, y and z be the coordinate functions that $\mathbf{S}^2$ inherits from its inclusion in $\mathbf{R}^3$. We will compute $\partial x/\partial u^\phi$.

For this we apply (4.14) with x_i replaced by u and the arbitrary function f specialized to the function x. This gives

$$\frac{\partial x}{\partial u^\phi}(m) = \frac{\partial \cos(u)\cos(v)}{\partial u}(\phi(m)) = -\sin(u^\phi(m))\cos(v^\phi(m)) \tag{4.14.3.1}$$

where u^ϕ and v^ϕ are the functions defined by $\phi(m) = (u^\phi(m), v^\phi(m))$.

Thus the function $\partial x/\partial u^\phi$ (4.14.1) is given by

$$m \mapsto -\sin(u^\phi(m))\cos(v^\phi(m))$$

We abbreviate this by writing

$$\frac{\partial x}{\partial u^\phi} = -\sin(u)\cos(v)$$

Similarly, we compute that

$$\begin{array}{lll} \dfrac{\partial x}{\partial u^\phi} = -\sin(u)\cos(v) & \dfrac{\partial y}{\partial u^\phi} = \cos(u)\cos(v) & \dfrac{\partial z}{\partial u^\phi} = 0 \\ \dfrac{\partial x}{\partial v^\phi} = -\cos(u)\sin(v) & \dfrac{\partial y}{\partial v^\phi} = -\sin(u)\sin(v) & \dfrac{\partial z}{\partial v^\phi} = \cos(v) \end{array} \tag{4.14.3.2}$$

Definition 4.15. Let $\phi : M \to N$ be a map of manifolds. Then for any point $m \in M$ we define a map

$$\phi_{*m} : T_mM \to T_{\phi(m)}N$$

by

$$\phi_{*m}(X)(df) = X(d(f \circ \phi))$$

Exercise 4.15.1. Show that ϕ_{*m} (4.15) is the result of applying the dual functor (I.3.2.2.3) to ϕ_m^* (4.7).

Exercise 4.15.2. If $N = \mathbf{R}^n$, show that

$$\phi_{*m}(X) = \sum_{j=1}^{n} X(\phi^*_{\phi(m)}(dx_j))\frac{\partial}{\partial x_j}$$

Show in particular that

$$\phi_{*m}\left(\frac{\partial}{\partial x_i^\phi}\right) = \frac{\partial}{\partial x_i}$$

Example 4.15.3. Consider the inclusion $i : \mathbf{S}^2 \hookrightarrow \mathbf{R}^3$. Fix a point $m \in S^2$. We will compute the action of i_{*m} on $T_m\mathbf{S}^2$.

First we need a coordinate patch containing m; we will assume that m is contained in the coordinate patch Ω of Example (4.14.3) and we will continue to use the notation of that example. Thus $\partial/\partial u^\phi$ and $\partial/\partial v^\phi$ form a basis for $T_m\mathbf{S}^2$ and we need to compute the action of i_{*m} on these vectors.

We will abuse notation by writing x, y and z both for the coordinate functions on $\mathbf{R}^3$ and for the coordinate functions $x \circ i$, $y \circ i$ and $z \circ i$ on $\mathbf{S}^2$. Note that x, y and z of (4.14.3) are actually $x \circ i$, $y \circ i$ and $z \circ i$.

Next we apply (4.15.2), specializing the arbitrary map ϕ (which is not the same as the chart ϕ that appears in this example!!) to the map $i : \mathbf{S}^2 \to \mathbf{R}^3$. This gives

$$\begin{aligned} i_{*m}\left(\frac{\partial}{\partial u^\phi}\right) &= \frac{\partial x}{\partial u^\phi}(m)\frac{\partial}{\partial x} \quad + \quad \frac{\partial y}{\partial u^\phi}(m)\frac{\partial}{\partial y} \quad + \quad \frac{\partial z}{\partial u^\phi}(m)\frac{\partial}{\partial z} \\ &= -\sin(u^\phi(m))\cos(v^\phi(m))\frac{\partial}{\partial x} + \cos(u^\phi(m))\cos(v^\phi(m))\frac{\partial}{\partial y} \end{aligned}$$

where we have used (4.14.3.1) to get the second equation.

We abbreviate this as

$$\begin{aligned} i_{*m}\left(\frac{\partial}{\partial u^\phi}\right) &= \frac{\partial x}{\partial u^\phi}\frac{\partial}{\partial x} \quad + \quad \frac{\partial y}{\partial u^\phi}\frac{\partial}{\partial y} \quad + \quad \frac{\partial z}{\partial u^\phi}\frac{\partial}{\partial z} \\ &= -\sin(u^\phi)\cos(v^\phi)\frac{\partial}{\partial x} + \cos(u^\phi)\cos(v^\phi)\frac{\partial}{\partial y} \end{aligned} \tag{4.15.3.1}$$

remembering that everything in sight is supposed to be evaluated at m.

Similarly, we write

$$
\begin{aligned}
i_{*m}\left(\frac{\partial}{\partial v^\phi}\right) &= \quad \frac{\partial x}{\partial v^\phi}\frac{\partial}{\partial x} \quad + \quad \frac{\partial y}{\partial v^\phi}\frac{\partial}{\partial y} \quad + \quad \frac{\partial z}{\partial v^\phi}\frac{\partial}{\partial z} \\
&= -\cos(u^\phi)\sin(v^\phi)\frac{\partial}{\partial x} - \sin(u^\phi)\sin(v^\phi)\frac{\partial}{\partial y} + \cos(v^\phi)\frac{\partial}{\partial z}
\end{aligned}
\tag{4.15.3.2}
$$

Proposition 4.16. Let

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

be maps of manifolds. Then for each $m \in M$

$$(\psi \circ \phi)_{*m} = \phi_{*\psi(m)} \circ \psi_{*m}$$

Corollary 4.16.1. If ϕ is a diffeomorphism, then ϕ_{*m} is an isomorphism of vector spaces.

Exercise 4.16.2. Use (4.15) and (4.16) to define a covariant functor T_0 that takes a vector space V to the tangent space $T_0 V$. (Here V is thought of as a manifold via (1.3.5).)

Exercises 4.16.3. Given a vector space V and elements $v, m \in V$, define an element $D_v \in T_m V$ by

$$D_v(df) = \lim_{t \to 0} \frac{f(m + tv)}{t}$$

i) Check that D_v is well-defined; in other words, check that if $df = dg$ then $D_v(df) = D_v(dg)$.

ii) Define a map

$$\begin{array}{rccc} \theta_{V,m} & : V & \to & T_m V \\ & v & \mapsto & D_v \end{array}$$

Show that θ_V is an isomorphism.

iii) Show that for $m = 0$, the maps $\theta_{V,0}$ constitute *natural* isomorphism from the identity functor to the functor T_0 defined in (4.16.2).

iv) Use (iii) to show that if V and W are vector spaces, then every linear transformation $T_0 V \to T_0 W$ is of the form f_{*0} for some $f : V \to W$.

v) If you've read the remarks in (4.8.2.1), generalize (iii) by replacing 0 with an arbitrary point $m \in V$ and defining naturality in the context of *pointed* vector spaces.

vi) Starting with the map $\xi_{V,m}$ of (4.9.3.2), apply the dual functor (I.3.4.3) to get an isomorphism

$$\tilde{\xi}_{V,m} : T_m V \to V^{**} \approx V$$

and show that $\theta_{V,m}$ is the inverse to $\tilde{\xi}_{V,m}$.

4.17. The Tangent Vector to a Curve. A *parameterized curve* on M is a smooth function $\gamma : I \to M$ for some open interval $I \subset \mathbf{R}$.

A parameterized curve is *imbedded* (or an *imbedded curve*) if

i) The map γ is injective and

ii) For every $a \in I$, the map $\gamma_{*a} : T_t I \to T_{\gamma(a)} M$ (4.12) is injective.

For a parameterized curve $\gamma : I \to M$ and a point $a \in I$, the *tangent vector to* γ *at* a, denoted $\gamma_*(a)$, is defined by

$$\gamma_*(a) = \gamma_{*a}\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)} M$$

where t is the standard coordinate function on $\mathbf{R}$.

For an imbedded curve, the *tangent vector to* γ *at* $\gamma(a) \in M$ is the same thing as the tangent vector to γ at a.

(Note that for an arbitrary parameterized curve, this last definition would make no sense, because we could have $\gamma(a) = \gamma(a')$ but $\gamma_*(a) \neq \gamma_*(a')$.)

Exercise 4.17.1. Let $\phi : U \to \mathbf{R}^n$ be a chart on M. Let $f : I \to \mathbf{R}^n$ be an imbedded curve such that

$$\mathrm{Image}(\hat{\gamma}) \subset \mathrm{Image}(\phi)$$

Set $\gamma = \phi^{-1} \circ f : I \to M$. Show that

$$\gamma_*(a) = \sum_{i=1}^{n} \frac{\partial x_i \circ f}{\partial t}(a) \frac{\partial}{\partial x_i^{\phi}}$$

where the x_i are the standard coordinate functions on $\mathbf{R}^n$.

In particular, for fixed i and fixed constants $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n$, consider the parameterized curve

$$\gamma : t \mapsto \phi^{-1}(\alpha_1, \ldots, \alpha_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n) \in M \tag{4.17.2}$$

where the domain I of γ is chosen so that (4.17.2) makes sense.

Then for any $a \in I$, $\gamma_*(a) = \partial/\partial x_i^\phi$ (4.11.4).

4C. Tensors

Definition 4.18. Let M be a manifold, m a point in M, and $T_m M$ the tangent space to M at m. Then the space of (p, q)-*tensors* at m is the vector space

$$T_m^{p,q} M = T^{p,q}(T_m M)$$

where the right-hand side is as defined in (I.5.2).

Let U be a coordinate patch containing m and $\phi : U \to \mathbf{R}^n$ a chart. Then by (4.10.1), (4.12) and (I.2.3.3), $T_m^{p,q} M$ has a basis consisting of all elements of the form

$$\frac{\partial}{\partial x_{i_1}^\phi} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_p}^\phi} \otimes dx_{j_i}^\phi \otimes \ldots \otimes dx_{j_q}^\phi \tag{4.18.1}$$

(This uses the identification (4.11.2.1).)

5. Cotangent Bundles, Tangent Bundles and Tensor Bundles

5A. The Cotangent Bundle.

Definition 5.1. Let M be a manifold. We will define a vector bundle called the *cotangent bundle* T^*M.

First, we define the set

$$T^*M = \{(m, \sigma) | m \in M, \sigma \in T_m^* M\}$$

Map $T^*M \xrightarrow{p} M$ by projection on the first factor.

We must define charts to make T^*M a manifold and a local trivialization to make T^*M a vector bundle.

For each chart (U, ϕ) on M, and for each point $u \in U \subset M$, let $\{dx_i^\phi(u)\}$ be the basis for T_m^*M described in (4.10.1) and (4.10.1.1). Then define a map

$$\begin{array}{cccc} \tau_\phi : & p^{-1}(U) & \to & U \times \mathbf{R}^n \\ & \left(u, \sum_{i=1}^n a_i dx_i^\phi(u)\right) & \mapsto & (u, a_1, \ldots, a_n) \end{array} \tag{5.1.1}$$

We want to show that as (U, ϕ) ranges over charts on M, the charts

$$\begin{array}{ccccccc} \psi_U : & p^{-1}(U) & \xrightarrow{\tau_\phi} & U \times \mathbf{R}^n & \to & \mathbf{R}^n \times \mathbf{R}^n & \approx \quad \mathbf{R}^{n^2} \\ & & & (m, x) & \mapsto & (\phi(m), x) & \end{array} \tag{5.1.2}$$

form an atlas, making T^*M a manifold, and the maps (5.1.1) form a local trivialization, making T^*M a vector bundle.

According to (2.20.1), it suffices to prove the following claim:

Claim 5.1.1. Let (V, ρ) be another chart on M. Then the map

$$\tau_\phi \circ \tau_\rho^{-1} : (U \cap V) \times \mathbf{R}^n \to (U \cap V) \times \mathbf{R}^n \tag{5.1.1.2}$$

is smooth.

Proof. We compute the map (5.1.1.2):

$$\begin{array}{rcll} (u, a_1, \ldots, a_n) & \overset{\tau_\rho^{-1}}{\longmapsto} & \left(u, \displaystyle\sum_{i=1}^n a_i dx_i^\rho(u)\right) & \\ & = & \left(u, \displaystyle\sum_{i=1}^n \sum_{j=1}^n a_i \frac{\partial x_i^\rho}{\partial x_j^\phi}(u) dx_j^\phi(u)\right) & \text{(by 4.14.2))} \\ & \overset{\tau_\phi}{\longmapsto} & \left(u, \displaystyle\sum_{i=1}^n a_i \frac{\partial x_i^\rho}{\partial x_1^\phi}(u), \ldots, \sum_{i=1}^n a_i \frac{\partial x_i^\rho}{\partial x_n^\phi}(u)\right) & \end{array}$$

and this is smooth by (1.11.1) and (4.14.1).

5B. The Tangent Bundle.

Definition 5.2. Let M be a manifold. Let TM be the dual (3.10) of the cotangent bundle (5.1). TM is called the *tangent bundle* of M.

Proposition 5.3. For $m \in M$, the fiber $(TM)_m$ is equal to (or at least naturally identified with) the tangent space $T_m M$.

Proof. This is an immediate consequence of construction (3.10) and remark (3.10.1).

Proposition 5.4. The dual of the tangent bundle is naturally isomorphic to the cotangent bundle.

Proof. This follows from (3.16).

5C. Tensor Bundles.

Definition 5.5. Let M be a manifold. Let $T^{p,q}M$ be the (p, q) tensor bundle (3.10) of the tangent bundle (5.1). $T^{p,q}M$ is called the *bundle of* (p, q)*-tensors on* M.

Proposition 5.6. For $m \in M$, the fiber $(T^{p,q}M)_m$ is equal to the tensor space $T^{p,q}_m M$.

Remark 5.6.1. It should be clear that $T^{1,0}M$ can be identified with the tangent bundle T_*M and $T^{0,1}M$ can be identified with the cotangent bundle T^*M.

6. One-Forms and Vector Fields

6A. One-Forms.

Definitions 6.1. Let M be a manifold. A *one-form* on M is a section (2.26) of the cotangent bundle T^*M.

Example 6.2. Let $f : M \to \mathbf{R}$ be a smooth function. Then for each $m \in M$, we get an element $df_m \in T^*_m M$ by (4.5.2). We can define a one-form $df : M \to T^*_m M$ by

$$df(m) = df_m$$

Exercise 6.2.1. Verify that df is really a section of the cotangent bundle; in particular, verify that it is a smooth map.

Example 6.3. Let $\mathbf{S}^1$ be the unit circle. Define $U_1 = \mathbf{S}^1 - (1,0)$ and define $\theta_i : U_i \to \mathbf{R}$ as in (1.3.6). For any $m \in U_i$, the function θ_i determines a cotangent vector $(d\theta_i)_m \in T^*_m M$. Moreover, on the set $U_1 \cap U_2$, $\theta_2 - \theta_1$ is constant, so for $m \in U_1 \cap U_2$ we have $(d\theta_1)_m = (d\theta_2)_m$. Thus we can unambiguously define a one-form

$$d\theta : \mathbf{S}^1 \to T^*M$$

by

$$d\theta(m) = (d\theta_i)_m \qquad \text{if } m \in U_i$$

6B. Vector Fields

Definition 6.4. A *vector field* on M is a section of the tangent bundle T_*M.

Example 6.5. Continue to use the notation of Example 6.3. In T_mM, there is a unique tangent vector $(\partial/\partial\theta)_m$ defined by the condition

$$\left(\frac{\partial}{\partial\theta}\right)_m (d\theta_m) = 1$$

and we can define a vector field $\partial/\partial\theta$ by

$$\left(\frac{\partial}{\partial\theta}\right)(m) = \left(\frac{\partial}{\partial\theta}\right)_m$$

Defintion 6.6. A manifold is *parallelizable* if its tangent bundle is trivial. A trivialization of T_*M is called a *parallelization* of M.

Proposition and Definition 6.7. An n- dimensional manifold M is parallelizable if and only if there are n vector fields $X_1, \ldots, X_n$ that are everywhere linearly independent on M. In that case we call the X_i a *global basis* for M.

Example 6.8. Let $\mathbf{S}^1$ be the unit circle. Then $\partial/\partial\theta$ (6.5) is a nowhere-zero vector field, so by (6.6), $\mathbf{S}^1$ is parallelizable.

(To see that $\partial/\partial\theta$ is nowhere zero, remember that $(\partial/\partial\theta)(m)$ is a linear map that takes $(d\theta)_m$ to 1, so it cannot be the zero map.)

Note that there are many other nowhere-zero vector fields on $\mathbf{S}^1$, any one of which suffices to demonstrate that $\mathbf{S}^1$ is parallelizable. Instead of the vector field $(\partial/\partial\theta)(m)$, we could have considered the vector field $f(m)(\partial/\partial\theta)(m)$ where f is any nowhere-zero real-valued map on $\mathbf{S}^1$.

Now let $\mathbf{S}^2$ be the two-sphere. It is possible to prove that there is not even one nowhere-zero vector field on $\mathbf{S}^2$. (A nowhere-zero vector field would amount to a continuous choice of a tangent vector at each point on the sphere; the fact that it is impossible to make such a choice is sometimes expressed by the saying "you can't come the hair on a ball".) Thus there is surely no hope of choosing two vector fields that are everywhere linearly independent, so $\mathbf{S}^2$ cannot be parallelizable.

Discussion 6.9. As m ranges over the manifold M, the tangent spaces T_mM are all distinct. Any two of these tangent spaces are isomorphic as vector spaces (just because they have the same dimension), but there is no preferred isomorphism between them and hence no way, even informally, to identify elements of one with elements of another.

If M is parallelizable, the choice of a parallelization fills that gap. Given an isomorphism

$$\phi : M \times \mathbf{R}^n \to T_*M$$

(in other words, given a parallelization), and given two points $m, m' \in M$, we get a preferred isomorphism

$$T_mM \to T_{m'}M$$

$$\phi(m, x) \mapsto \phi(m', x)$$

which, loosely speaking, allows us to identify T_mM with $T_{m'}M$.

By construction, every point $m \in M$ is contained in an open set U such that U (thought of as a manifold in its own right) is parallelizable. So one can always find a way to identify the tangent space at m with tangent spaces at neighboring points. But it is important to realize that such identifications depend on the choice of parallelization; a different parallelization yields a different set of identifications.

6C. One-Parameter Groups.

You can think of a vector field as a way of (smoothly) placing a tangent vector at each point of M. Once you've done this, you can imagine starting at an arbitrary point of M and constructing a curve by "following the tangent vectors", i.e. keeping the curve tangent to the given vectors at each point. You can then start at another arbitrary point and do the same thing, and attempt to continue until you have completely filled M with curves. The theory of one- parameter groups describes the extent to which this is possible.

It turns out the opposite problem— starting with the curves and constructing the vector field—is easier, so we'll begin with that; then we'll turn to the problem of starting with the vector field and constructing the curves.

Definition 6.10. A *one- parameter group* on a manifold M is a smooth map

$$g : M \times \mathbf{R} \to M$$

such that for all $m \in M$ and $s, t \in \mathbf{R}$ we have

$$g(m, 0) = x$$

$$g(g(m, s), t) = g(m, s + t)$$

Thus for each fixed x, the map

$$t \mapsto g(m, t) \tag{6.10.1}$$

is a parameterized curve (4.17) on M. The images of any two of these parameterized curves are either identical or disjoint (prove!), so you can think of a one- parameter group as "filling M with disjoint curves".

Given a one-parameter group g and a point $m \in M$, let $X(m)$ be the tangent vector at m to the curve (6.10.1). (See (4.17) for the definition of the tangent vector to a parameterized curve.) Then the map $m \mapsto X(m)$ is a vector field, which we call the *vector field associated to the one- parameter group g.*

Definition 6.11. The natural question now is: Does every vector field arise from a one-parameter group? The answer is: not quite. To make this precise, we need to define a *local one-parameter group*. Let U be an open subset of M. Then a *local one-parameter group on* U consists of an open interval $I \subset \mathbf{R}$ containing 0, and a map

$$g : U \times I \to M$$

satisfying the conditions of (6.10) whenever s, t and $s + t$ are all in I.

A local one-parameter group defines a family of parameterized curves just as before, and hence defines a vector field on U via $m \mapsto X(m)$ where $X(m)$ is the tangent vector at m to the curve given by equation (6.10.1).

Facts 6.12. Given a vector field X and a point $m \in M$, there exists an open set U containing m and a local one- parameter group g on U such that the restriction of X to U is the vector field associated to g.

In other words, every point in m is surrounded by an open neighborhood that can be "filled with curves" whose tangent vectors are those prescribed by X.

Moreover, g is unique in the following sense: if h is another local one- parameter group defined on an open set U' and an open interval I', and if g and h both give rise to the same vector field X on $U \cap U'$, then the restrictions of g and h to $(U \cap U') \times (I \cap I')$ are equal.

To prove these facts, one first uses coordinate charts to reduce to the case $M = \mathbf{R}^n$ and then uses standard existence and uniqueness theorems from the theory of differential equations.

Definition 6.13. The parametrized curves $t \mapsto g(m, t)$ (for fixed m) are called *integral curves* or *flow lines* for the corresponding vector field. Another way to say this is: A parameterized curve $\gamma : I \to M$ is an integral curve for X if, for every $m \in \gamma(I)$, $X(m)$ is the tangent vector to $\gamma(m)$.

Exercise 6.14. Consider the vector field $\frac{\partial}{\partial x}$ on $\mathbf{R}^2$. Show that the integral curves for this vector field are all straight lines.

6D. Derivations.

Definition 6.15. Let M be a manifold and let $C(M)$ be the vector space of all smooth functions on M. A linear transformation $\partial : C(M) \to C(M)$ is called a *derivation* if it satisfies

$$\partial(fg) = f\partial(g) + g\partial(f)$$

6.16. The derivation associated to a vector field. Let $X : M \to TM$ be a vector field. We define a derivation ∂_X as follows: Given $f \in C(M)$ and given $m \in M$, let $df_m \in T_m^*M$ be the cotangent vector associated with f (4.5.2). Now set

$$\partial_X(f)(m) = X(m)(df_m) \tag{6.16.1}$$

Abuse of Notation 6.16.2. When X is a vector field, we will sometimes use the same symbol X to denote the derivation ∂_X. Thus if f is a smooth function, $X(f)$ is another smooth function and it is defined by the equation

$$X(f)(m) = X(m)(df_m)$$

Proposition 6.17. Every derivation is the derivation associated to some vector field.

Proof. Given a derivation $\partial : C(M) \to C(M)$, given a point $m \in M$ and given a cotangent vector $df_m \in T_m^*M$, set

$$X(m)(df_m) = \partial(f)(m) \tag{6.17.1}$$

Assuming for the moment that this is well- defined (in other words, assuming that the right-hand side of (6.17.1) is unambiguous), $X(m) : T_m^*M \to \mathbf{R}$ is a linear transformation, so $X(m) \in T_mM$; in other words X is a section of TM, as desired. (You should check smoothness!)

As for well-definedness, we have to verify that if $df_m = dg_m$ then $\partial(f)(m) = \partial(g)(m)$; in the notation of (4.5), this means we have to check that $X(m)$ vanishes on $\mathcal{M}_m^2$. In other words, we need to know that if $\xi_i(m) = \rho_i(m) = 0$ for all i, then

$$\partial\left(\sum_{i=1}^{n} \xi_i\rho_i\right)(m) = 0$$

But this follows immediately from the linearity of ∂ and (6.16.1).

Proposition 6.18. If X and Y are vector fields, then the map

$$(X \circ Y - Y \circ X) : C(M) \to C(M)$$

is a derivation.

(Here we are treating identifying X and Y with the corresponding derivations ∂_X and ∂_Y per (6.16.2), and $\circ$ is composition of functions.)

Proof. Linearity is immediate; we have to check (6.16.1). We have

$$\begin{aligned}(X \circ Y)(fg) &= X(fY(g) + gY(f)) \\ &= fX(Y(g)) + X(f)Y(g) + gX(Y(f)) + X(g)Y(f)\end{aligned}$$

and similarly

$$(Y \circ X)(fg) = fY(X(g)) + Y(f)X(g) + gY(X(f)) + Y(g)X(f)$$

Subtracting gives the needed equation:

$$(X \circ Y - Y \circ X)(fg) = f(X \circ Y - Y \circ X)(g) + g(X \circ Y - Y \circ X)(f)$$

Remark 6.18.1. Although the difference $X \circ Y - Y \circ X$ is a derivation, it is usually not the case that either $X \circ Y$ or $Y \circ X$ is a derivation on its own.

Definition 6.19. Let X and Y be vector fields. By (6.18) and (6.17), we have

$$X \circ Y - Y \circ X = Z$$

for some vector field Z. We call Z the *Lie bracket* of X and Y and we write

$$Z = [X, Y]$$

Exercise 6.20. Let $M = \mathbf{R}^n$ and let $x_1, \ldots, x_n$ be the coordinates on $\mathbf{R}^n$. Show that

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

for every i and j.

Exercise 6.21. Let M be a manifold and let $\phi : M \to \mathbf{R}^n$ be a diffeomorphism. (For example, (M, ϕ) could be a coordinate patch on a larger manifold.) Show that

$$\left[\frac{\partial}{\partial x_i^\phi}, \frac{\partial}{\partial x_j^\phi}\right] = 0$$

for every i and j.

Exercise 6.22. Let $M = \mathbf{R}^2$ with coordinates x and y. Let

$$X = cos(x^2 + y^2)\frac{\partial}{\partial x} + sin(x^2 + y^2)\frac{\partial}{\partial y}$$

$$Y = -sin(x^2 + y^2)\frac{\partial}{\partial x} + cos(x^2 + y^2)\frac{\partial}{\partial y}$$

Show that

$$[X, Y] = -2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

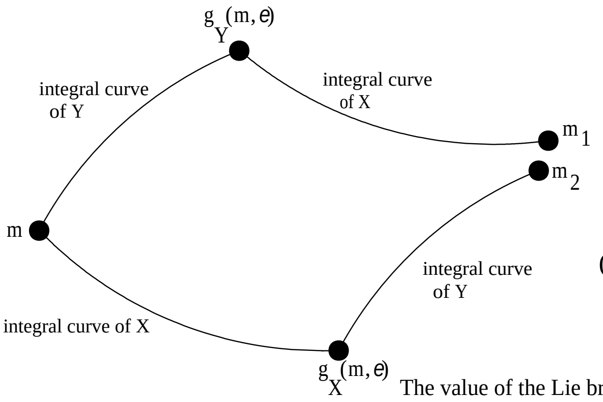
Exercise 6.23. Let X, Y, and Z be vector fields on M and let $\phi : M \to \mathbf{R}$ be a smooth function. Show that

$$[\phi X + Y, Z] = \phi[X, Z] + [Y, Z] - Z(\phi)X$$

$$[X, \phi Y + Z] = \phi[X, Y] + [X, Z] - X(\phi)Y$$

Discussion 6.24. The Lie bracket measures the failure of the vector fields X and Y to commute. Here is a geometric interpretation of that failure:

Let $m \in M$. Imagine pushing m a short distance along an integral curve of Y and then a short distance along an integral curve of X. This gives a point m_1. Alternatively, push m a short distance along an integral curve of X and then a short distance along an integral curve of Y. This gives a point m_2. The bracket $[X, Y]$ measures the failure of the points m_1 and m_2 to coincide. (See Figure 6.24.1.)



(6.24.1)

The value of the Lie bracket [X,Y] at m measures the failure of the "quadrilateral" to close.

More precisely, let g_X and g_Y be the one-parameter groups associated with X and Y. Let $m \in M$, let $\epsilon > 0$ be a small real number, and define

$$m_1 = g_X(g_Y(m, \epsilon), \epsilon)$$

$$m_2 = g_Y(g_X(m, \epsilon), \epsilon)$$

The failure of X and Y to commute is measured by the "difference" between m_1 and m_2. But how can we measure this difference? What are the appropriate units? Here's the trick: Let $f : M \to \mathbf{R}$ be any smooth function at all and look at $f(m_1) - f(m_2)$. Of course, this difference can have any value (because f is completely arbitrary). But we can still talk meaningfully about how this value varies with the choice of f and with the choice of ϵ (which it does, because the points m_1 and m_2 depend on ϵ). So put

$$F(\epsilon) = f(m_1) - f(m_2)$$

and note for example that $F(0) = 0$. Then the derivative $F'(0)$ is a measure of how quickly m_1 and m_2 spread out from each other as ϵ increases from zero, and this measure depends on the choice of f via the formula

$$F'(0) = [X, Y](f)$$

Roughly, then, $[X, Y]$ accounts for that part of the infinitesimal discrepancy between $f(m_1)$ and $f(m_2)$ that does not depend on f.