0. Executive summary: The Ramsey rule gives optimal tax rates when only a subset of goods can be taxed. The formula involves compensated cross-elasticities in a complicated way. When the subset of taxable goods is equal to the entire set of goods, the Ramsey rule reduces to proportional taxation, which of course we already knew was optimal.

1. Suppose $n+m$ goods, all supplied inelastically. Good i is supplied at price P_i. The first n goods can be taxed; the remaining m goods can't be. Suppose also that labor is supplied inelastically, or equivalently that income I is exogenous.

2. Let U be the utility function. For taxes $T_1, \ldots, T_n,$, let $X_i(T_1, \ldots, T_n)$ be the quantity demanded of good i.

3. The tax vector is efficient if it maximizes the expression

$$\sum_{i=1}^{n} T_i X_i$$

subject to holding fixed the value of $U(X_1, \ldots, X_n)$

4. The first-order conditions for efficiency are

$$\sum_j T_j \frac{\partial X_j}{\partial T_i} + X_i = \sum \lambda U_j \frac{\partial X_j}{\partial T_i} \qquad (i = 1, \ldots, n)$$

5. By individal maximization, U_j is proportional to $P_j + T_j$, and by differentiating the budget constraint we have

$$\sum (P_j + T_j) \frac{\partial X_j}{\partial T_i} = -X_i$$

. Thus the right-hand side in paragraph 4 is proportional to X_i.

6. Let M be the matrix of partials $\left(\frac{\partial X_j}{\partial X_i}\right)$. Let T be the column vector of taxes and let X be the column vector of quantities. Then paragraphs 4 and 5 show that T is proportional to

$$M^{-1} \cdot X$$

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7. The case of two taxed goods: We write X and Y instead of X_1 and X_2. Paragraph 6 implies that the optimal tax rates on goods X and Y are inversely proportional to

$$\eta_{X,X} - \frac{P_X Q_X}{P_Y Q_Y} \eta_{XY}$$

$$\eta_{Y,Y} - \frac{P_Y Q_Y}{P_X Q_X} \eta_{YX}$$

Here "tax rate" means T_i/P_i and $\eta_{X,Y}$ is the compensated elasticity of demand for X with respect to the price of Y.

8. The case of three taxed goods: We write X, Y and Z for the goods. The conditions in paragraph 4 imply that the optimal tax rates on X and Y should be inversely proportional to

$$\eta_{YY}(\eta_{XZ} - \eta_{ZZ}) + \eta_{XY}(\eta_{ZZ} - \eta_{YZ}) + \eta_{ZY}(\eta_{YZ} - \eta_{XZ}$$

$$\eta_{XX}(\eta_{YZ} - \eta_{ZZ}) + \eta_{YX}(\eta_{ZZ} - \eta_{XZ}) + \eta_{ZX}(\eta_{XZ} - \eta_{YZ})$$

and similarly for the ratio of the tax rates on X and Z or Y and Z.

9. The case of n taxed goods: The expression will be a polynomial of degree $n-1$ in the various cross-elasticities; I don't have a clever way of writing it.

10. **Theorem.** If all goods are taxed, then the optimal tax rates are all identical.

Proof. Differentiating the budget constraint shows that the vector $(P+T)$ is proportional to

$$M^{-1} \cdot X$$

and hence, because M is of corank 1, is proportional to T by paragraph 6.

EDITED TO ADD: Paragraphs 5, 6 and 7 assume no cross-elasticities between taxed and untaxed goods. This assumption is of course automatically true in paragraph 10.