Some aspects of homological algebra

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September 10, 2010

[1]The essential content of Chapters 1, 2, and 4, and part of Chapter 3 was developed in the spring of 1955 during a seminar in homological algebra at the University of Kansas. Received March 1, 1957.

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Introduction

0.1 Content of the article.

This works originates from an attempt to take advantage of the formal analogy between the cohomology theory with of a space with coefficients in a sheaf [4, 5] and the theory of derived functors of a functor on a category of modules [6], in order to find a common framework to encompass these theories and others.

This framework is sketched in Chapter 1, whose theme is the same as that of [3]. These two expositions do not overlap, however, except in 1.4. I have particularly wished to provide usable criteria, with the aid of the concepts of infinite sums and products in abelian categories, for the existence of sufficiently many injective or projective objects in abelian categories, without which the essential homological techniques cannot be applied. In addition, for the reader's convenience, we will give a thorough exposition of the functorial language (1.1, 1.2, and 1.3). The introduction of additive categories in 1.3 as a preliminary to abelian categories provides a convenient language (for example to deal with spectral functors in Chapter 2).

Chapter 2 sketches the essential aspects of homological formalism in abelian categories. The publication of [6] has allowed me to be very concise, given that the Cartan-Eilenberg techniques can be translated without change into the new context. Sections 2.1 and 2.2 however, were written so as not to exclude abelian categories that do not contain sufficiently many injectives or projectives. Later sections are based on resolutions, employing the usual techniques. Sections 2.4 and 2.5 contain a variety of additional material and are essential for understanding what follows them. In particular, Theorem 2.4.1 gives a mechanical method for obtaining most known spectral sequences (or, in any case, all those encountered in this work).

In Chapter 3, we redevelop the cohomology theory of a space with coefficients in a sheaf, including Leray's classical spectral sequences. The treatment provides additonal flexibility compared with [4, 15], in particular, given that all the essential results are found without any restrictive hypotheses on the relevant spaces, either in this chapter or any later one, so that the theory also applies to the non-separated spaces that occur in abstract algebraic geometry or in arithmetical geometry [15, 8]. Conversations with Roger Godement and Henri Cartan were very valuable for perfecting the theory. In particular, Godement's introduction of flabby sheaves and soft sheaves, which can useful be substituted for fine sheaves in many situations, has turned out to be extremely convenient. A more complete description, to which we will turn for a variety of details, will be given in a book by Godement in preparation [9].

Chapter 4 deals with the non-classical question of Ext of sheaves of modules; in particular, it contains a useful spectral sequence that relates global and local Ext. Things get more complicated in Chapter 5, in which, in addition, a group G operates on the space X, the sheaf $\mathbf{O}$ of rings over X and the sheaf of $\mathbf{O}$-modules under consideration. Specifically, in 5.2, we find what seems to me to be the definitive form of the Čech cohomology theory of spaces acted on, possibly with fixed points, by an abstract group. It is stated by introducing new functors $H^n(X; G, A)$ (already implicit in earlier specific cases); we then find two spectral functors with remarkable initial terms that converge to it.

0.2 Applications

In this article, for want of space, I have been able to provide only very few applications of the techniques used (mainly in 3.4 and 3.6), restricting myself to noting only a few in passing. We indicate the following applications.

(a) The notion of Ext of sheaves of modules allows the most general formulation known of Serre's algebraic duality theorem: If A is a coherent algebraic sheaf [15] on a projective algebraic variety of dimension n without singularities, then the dual of $H^p(X, A)$ is canonically identified with $\mathrm{Ext}^{n-p}_{\mathbf{O}}(X; A\Omega^n)$, where $\mathbf{O}$ (respectively Ω^n) is the sheaf of germs of regular functions (respectively, of regular n-forms) over X.

(b) All the formalism developed in Chapters 3, 4, and 5 can apply to abstract algebraic geometry. I will show elsewhere how it makes possible the extension of various results proved by Serre [15, 16, 17] for projective varieties, to complete algebraic varieties.

(c) It seems that the $H^n(X; G, A)$ are the natural intermediaries for a general theory of reduced Steenrod powers in sheaves, and the cohomology of symmetric powers of arbitrary spaces, a theory which also applies in algebraic geometry in characteristic p.

0.3 Omissions

To avoid making this memoir overly long, I have said nothing about questions on multiplicative structures, although they are essential for applying the concepts in Chapters 3, 4, and 5. Note, moreover, that there does not yet seem to be any satisfactory theory of multiplicative structures in homological algebra that have the necessary generality and simplicity ([6, Chapter II] being a striking illustration of this state of affairs).[1′] For multiplication in sheaf cohomology a satisfactory description can be found in [9]. The reader will notice numerous other omissions.

I am happy to express my thanks to Roger Godement, Henri Cartan, and Jean-Pierre Serre, whose interest was the indispensable stimulus for the writing of this memoir.

[1′] Pierre Cartier has recently found a satisfactory general formulation for multiplicative structures in homological algebra which he will announce himself.

Translator's preface

We found and fixed (mostly silently) innumerable errors in the text and doubtless introduced many of our own. In a number of cases, we have simplified Grothendieck's somewhat tortured sentences that sometimes went on interminably with parenthetical inserts. In a few cases, we have updated the language (for example, replacing "functor morphism" by "natural transformation"). In one or two places, we were unable to discern what he meant.

One curiosity is that Grothendieck seems to have had an aversion to the empty set. Products and sums are defined only for non-empty index sets and even finitely generated modules are required to have at least one non-zero generator. The zero module is not considered free (although it is, obviously, finitely generated). Except that his definition of complete is incomplete, this aversion does not really affect anything herein.

Grothendieck treats a category as a class of objects, equipped with a class of morphisms. This differs from both the original view expressed in Eilenberg and Mac Lane[a] and in later and current views, in which a category consists of both the objects and arrows (or even of the arrows alone, since the objects are recoverable). This shows up in several ways, not least that he writes $A \in \mathbf{C}$ to mean that A is an object of $\mathbf{C}$ and, most importantly, he says "$\mathbf{C}$ is a set" to say what we would express as "$\mathbf{C}$ has a set of objects".

One point to be made is that Grothendieck systematically uses "$=$" where we would always insist on "$\cong$". The structualists who founded Bourbaki wanted to make the point that isomorphic structures should not be distinguished, but category theorists now recognize the distinction between isomorphism and equality. For example, all of Galois theory is dependent on the automorphism group which is an incoherent notion in the structuralist paradigm. For the most part, we have replaced equality by isomorphism, when it seems appropriate.

These comments would be incomplete without a word about copyright issues. We do not have Grothendieck's permission to publish this. His literary executor, Jean Malgoire refused to even ask him. What we have heard is that Grothendieck "Does not believe in" copyright and will have nothing to do with it, even to release it. So be it. We post this at

[a]Translator's note: S. Eilenberg and S. Mac Lane, General theory of natural equivalences. Trans. A. M. S. 58, 1945, 231–244.

our peril and you download it, if you do, at yours. It seems clear that Grothendieck will not object, while he is alive, but he has children who might take a different view of the matter.

Despite these comments, the carrying out of this translation has been an interesting, educational, and enjoyable activity. We welcome comments and corrections and will consider carefully the former and fix the latter.

Update, March, 2010. Since the above was written in Dec. 2008, there has been a new development. Grothendieck has asked that all republication of any of his works (in original or translation) be ended. He has not actually invoked copyright (which, as stated above, he does not believe in), but asked this as some sort of personal privilege. This makes no sense and Grothendieck never expressed such a wish before. I personally believe that Grothendieck's work, as indeed all mathematics including my own modest contributions, are the property of the human race and not any one person. I do accept copyright but only for a very limited time. Originally in the US, copyright was for seven years, renewable for a second seven. These periods were doubled and then doubled again and the copyright has now been extended essentially indefinitely, without the necessity of the author's even asking for a copyright or extension. This is a perversion of the original purpose of copyright, which was not to make intellectual achievements a property, but rather to encourage the publication, eventually into the public domain, of creative efforts.

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Marcia L. Barr

Michael Barr

Chapter 1

Generalities on abelian categories

1.1 Categories

Recall that a *category* consists of a non-empty class $\mathbf{C}$ of objects[b] together with, for $A,\ B \in \mathbf{C}$, a set $\mathrm{Hom}(A, B)$ collectively called morphisms of A into B, and for three objects $A,\ B,\ C \in \mathbf{C}$ a function (called composition of morphisms) $(u, v) \mapsto vu$ of $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$, which satisfy the following two axioms: the composition of morphisms is *associative*; for $A \in \mathbf{C}$, there is an element $i_A \in \mathrm{Hom}(A, A)$ (called the identity morphism of A) which is a right and left unit for the composition of morphisms. (The element i_A is then unique.) Finally, it will be prudent to suppose that a morphism u determines its source and target. In other words, if A, B and A', B' are two distinct pairs of objects of $\mathbf{C}$, then $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ are disjoint sets.

If $\mathbf{C}$ is a category, we define the *dual category* $\mathbf{C}^o$ as the category with the same objects as $\mathbf{C}$, and where the set $\mathrm{Hom}(A, B)^o$ of morphisms of A into B is identical to $\mathrm{Hom}(B, A)$, with the composite of u and v in $\mathbf{C}^o$ being identified as the composite of v and u in $\mathbf{C}$. Any concept or statement about an arbitrary category admits a dual concept or statement (the process of reversing arrows), which will be just as useful in the applications. Making this more explicit is usually left to the reader.

Suppose we are given a category $\mathbf{C}$ and a morphism $u : A \longrightarrow B$ in $\mathbf{C}$. For any $C \in \mathbf{C}$, we define a function $v \mapsto uv : \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, B)$ and a function $w \mapsto wu : \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$. We say that u is a *monomorphism* or that u is *injective* (respectively, u is an *epimorphism* or u *is surjective*) if the first (respectively, the second) of the two preceding functions is always injective; u is called *bijective* if u is both injective and surjective. We call a *left inverse* (respectively, a *right inverse*) of u a $v \in \mathrm{Hom}(B, A)$

[b]Translator's note: Modern usage would move the $\mathbf{C}$ to before the word "consists". That is, a category consists of both the objects and the morphisms. Also we allow the existence of an empty category, having no objects and no arrows.

such that $vu = i_A$ (respectively $uv = i_B$); v is called *the inverse of* u if it is both a left inverse and a right inverse of u (in which case it is uniquely determined). u is called an *isomorphism* if it has an inverse. If u has a left inverse (respectively, a right inverse) it is injective (respectively surjective). Thus an isomorphism is bijective (the converse being, in general, false).

The composite of two monomorphisms (respectively, epimorphisms) is a monomorphism (respectively, epimorphisms), hence the composite of two bijections is a bijection; similarly the composite of two isomorphisms is an isomorphism. If the composite vu of two morphisms u, v is a monomorphism (respectively, an epimorphism), then u (respectively, v) is as well. Although the development of such trivialities[c] is clearly necessary, we will subsequently refrain from setting them forth explicitly, feeling it sufficient to indicate the definitions carefully.

Consider two monomorphisms $u : B \longrightarrow A$ and $u' : B' \longrightarrow A$. We say that u' *majorizes* or *contains* u and we write $u \leq u'$ if we can factor u as $u'v$ where v is a morphism from B to B' (which is then uniquely determined). That is a *preorder* in the class of monomorphisms with target A. We will say that two such monomorphisms u, u' are *equivalent* if each one contains the other. Then the correponding morphisms $B \longrightarrow B'$ and $B' \longrightarrow B$ are inverse isomorphisms. Choose (for example, using Hilbert's all-purpose symbol τ) a monomorphism in each class of equivalent monomorphisms: the selected monomorphisms will be called *subobjects*[d] of A. Thus a subobject of A is not simply an object of $\mathbf{C}$, but an object B, together with a monomorphism $u : B \longrightarrow A$ called the *canonical injection* of B into A. (Nonetheless, by abuse of language, we will often designate a subobject of A by the name B of the corresponding object of $\mathbf{C}$.) The containment relation defines an *order* relation (not merely a preorder relation) on the class of subobjects of A. It follows from the above that the subobjects of A that are contained in a subobject B are identified with the subobjects of B, this correspondence respecting the natural order[e]. (This does not mean, however, that a subobject of B is *equal* to a subobject of A, which would require $A = B$.)

Dually, consideration of a preorder on the class of epimorphisms of A makes it possible to define the ordered class of *quotient objects* of A.

Let $A \in \mathbf{C}$ and let $(u_i)_{i \in I}$ be a non-empty family of morphisms $u_i : A \longrightarrow A_i$. Then for any

[c]Translator's note: Grothendieck used the word "sorites" here, apparently in the sense of a connected string of inferences

[d]Translator's note: Grothendieck used the word "sous-truc", but modern usage has hardened on "subobject"

[e]Translator's note: When we read this, we first thought that Grothendieck was making a claim, which would have been false, that in any category, it would be possible to choose a subobject in each equivalance class of monomorphisms in such a consistent way that a subobject of a subobject was a subobject. But, having seen how Grothendieck uses "identified with" later on, I now think he is making no such claim.

$B \in \mathbf{C}$, the functions $v \mapsto u_i v : \mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(B, A_i)$ define a natural transformation

$$\mathrm{Hom}(B, A) \longrightarrow \prod_{i \in I} \mathrm{Hom}(B, A_i)$$

We say that the u_i define a *representation of A as a direct product of the A_i* if for any B, the preceding displayed function is bijective. If this holds and if A' is another object of $\mathbf{C}$ represented as a product of the A'_i by morphisms $u'_i : A' \longrightarrow A'_i$ (the set of indices being the same), then for any family (v_i) of morphisms $v_i : A_i \longrightarrow A'_i$, there is a unique morphism $v : A \longrightarrow A'$ such that $u'_i v = v_i u_i$ for all $i \in I$. From this we conclude that if the v_i are equivalences, this holds for v: in particular, if the v_i are identity I_{A_i} we see that two objects A, A' represented as products of the family A_i are canonically isomorphic. It is therefore natural to select among all the $(A, (u_i))$, as above, a particular system, for example using Hilbert's τ symbol that will be called the *product* of the family of objects $(A_i)_{i \in I}$. It is therefore not a simple object A of $\mathbf{C}$, but such an object equipped with a family (u_i) of morphisms to A_i, called the *canonical projections from the product* to its factors A_i. We indicate the product of the A_i (if it exists) by $\prod_{i \in I} A_i$. If I is reduced to a single element i, then the product can be identified with A_i itself. We say that $\mathbf{C}$ is a *category with products* if the product of two objects of $\mathbf{C}$ always exists (then it holds for the product of any non-empty finite family of objects of $\mathbf{C}$).[f] We say that $\mathbf{C}$ is a *category with infinite products* if the product of any non-empty family of objects of $\mathbf{C}$ always exists. We have seen that if there are two products, $A = \prod_{i \in I} A_i$ and $B = \prod_{i \in I} B_i$ corresponding to the same set I of indices, then a family (v_i) of morphisms A_i to B_i canonically defines a morphism v from A to B, called *product of the morphisms* v_i and sometimes denoted $\prod_{i \in I} v_i$. If the v_i are monomorphisms, so is their product. But the analogous statement for epimorphisms fails in general (as we see, for example, in the category of sheaves over a fixed topological space).

Dual considerations of the preceding can be used to define the notions of a *representation of an object as a sum* of a family of objects A_i by means of morphisms $u_i : A_i \longrightarrow A$ (for any $B \in \mathbf{C}$, the natural transformation

$$\mathrm{Hom}(A, B) \longrightarrow \prod_{i \in I} \mathrm{Hom}(A_i, B)$$

is bijective), of a *direct sum* $\bigoplus_{i \in I} A_i$, equipped with *canonical injections* $A_i \longrightarrow \bigoplus_{i \in I} A_i$ (which, however, are not necessarily monomorphisms, despite their name), and also equipped with with a *sum morphism* $\bigoplus_{i \in I} u_i$ of a family of morphisms $u_i : A_i \longrightarrow B_i$. If the u_i are epimorphisms, their sum is as well.

[f] Translator's note: Nowadays, we would not say that a category has products unless it has products over all index sets I, including the empty set, so the definition does not even define finite products.

1.2 Functors

Let $\mathbf{C}$ and $\mathbf{C}'$ be categories. Recall that a *covariant functor* from $\mathbf{C}$ to $\mathbf{C}'$ consists of a "function" F which associates an object $F(A) \in \mathbf{C}'$ to each $A \in \mathbf{C}$ and a morphisms $F(u) : F(A) \longrightarrow F(B)$ in $\mathbf{C}'$ to each morphism $u : A \longrightarrow B$ in $\mathbf{C}$, such that we have $F(I_A) = I_{F(A)}$ and $F(vu) = F(v)F(u)$. There is an analogous definition of *contravariant functor* from $\mathbf{C}$ to $\mathbf{C}'$ (which are also covariant functors from $\mathbf{C}^o$ to $\mathbf{C}'$ or from $\mathbf{C}$ to $\mathbf{C}'^o$). We similarly define functors of several variables (or *multifunctors*), covariant in some variables and contravariant in others. In order to simplify, we will generally limit ourselves to functors of one variable. Functors are composed in the same way as functions are, this composition is associative and "identify functors" play the role of units.

Let $\mathbf{C}$ and $\mathbf{C}'$ be fixed categories, F and G covariant functors from $\mathbf{C}$ to $\mathbf{C}'$. A *functorial morphism* f from F to G (also called a "natural transformation" from F to G by some authors[g]) is a "function" that associates a morphism $f(A) : F(A) \longrightarrow G(A)$ to any $A \in \mathbf{C}$, such that for any morphism $u : A \longrightarrow B$ in $\mathbf{C}$ the following diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(u)} & F(B) \\ {\scriptstyle f(A)}\downarrow & & \downarrow{\scriptstyle G(u)} \\ G(A) & \xrightarrow[f(B)]{} & G(B) \end{array}$$

commutes. Natural transformations $F \longrightarrow G$ and $G \longrightarrow H$ are composed in the usual way. Such composition is associative, and the "identity transformation" of the functor F is a unit for composition of natural transformations. (Therefore, if $\mathbf{C}$ is a set[h] the functors from $\mathbf{C}$ to $\mathbf{C}'$ again form a category.) Note that the composite GF of two functors $F : \mathbf{C} \longrightarrow \mathbf{C}'$ and $G : \mathbf{C}' \longrightarrow \mathbf{C}''$ is, in effect, a bifunctor with respect to the arguments G and F: a natural transformation $G \longrightarrow G'$ (respectively, $F \longrightarrow F'$) defines a natural transformation $GF \longrightarrow G'F$ (respectively, $GF \longrightarrow GF'$).

An *equivalence* of a category $\mathbf{C}$ with a category $\mathbf{C}'$ is a system (F, G, ϕ, ψ) consisting of covariant functors:

$$F : \mathbf{C} \longrightarrow \mathbf{C}' \qquad G : \mathbf{C}' \longrightarrow \mathbf{C}$$

[g] Translator's note: It is now called a "natural transformation" by all authors and this phrase will be used subsequently

[h] Translator's note: He means that $\mathbf{C}$ should be small, not that it be discrete.

and of natural equivalences[i]

$$\phi : 1_{\mathbf{C}} \longrightarrow GF \qquad \psi : 1_{\mathbf{C}'} \longrightarrow FG$$

(where $1_{\mathbf{C}}$ and $1_{\mathbf{C}'}$ are the identity functors of $\mathbf{C}$ and $\mathbf{C}'$, respectively) such that for any $A \in \mathbf{C}$ and $A' \in \mathbf{C}'$, the composites

$$F(A) \xrightarrow{F(\phi(A))} FGF(A) \xrightarrow{\psi^{-1}(F(A))} F(A)$$

$$G(A') \xrightarrow{G(\psi(A'))} GFG(A') \xrightarrow{\phi^{-1}(G(A'))} G(A')$$

are the identities of $F(A)$ and $G(A')$, respectively. Then for any pair A, B of objects of $\mathbf{C}$, the functions $f \mapsto F(f)$ from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(F(A), F(B)$ is a bijection whose inverse is the function $g \mapsto G(g)$ from $\mathrm{Hom}(F(A), F(B))$ to $\mathrm{Hom}(GF(A), GF(B))$, which is identified with $\mathrm{Hom}(A, B)$ thanks to the isomorphisms $\phi(A) : A \longrightarrow GF(A)$ and $\phi(B) : B \longrightarrow GF(B)$. Equivalences between categories compose like functors. Two categories are called *equivalent* if there is an equivalence between them. Current usage will not distinguish between equivalent categories. It is important, however, to observe the difference between this notion and the stricter notion of isomorphism (which applies if we wish to compare categories that are sets). Let $\mathbf{C}$ be a non-empty set. For any pair of objects A, $B \in \mathbf{C}$, suppose that $\mathrm{Hom}(A, B)$ consists of one element. Then under the unique composition laws $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$, $\mathbf{C}$ becomes a category, and two categories constructed by this procedure are always equivalent, but they are isomorphic only if they have the same cardinality. None of the equivalences of categories that we encounter in practice is an isomorphism.

1.3 Additive categories

An *additive category* is a category $\mathbf{C}$ for which is given, for any pair A, B of objects of $\mathbf{C}$ an abelian group law in $\mathrm{Hom}(A, B)$ such that the composition of morphisms is a bilinear operation. We suppose also that the sum and the product of any two objects A, B of $\mathbf{C}$ exist. It is sufficient, moreover, to assume the existence of the sum *or* the product of A and B exists; the existence of the other can be easily deduced and, in addition, $A \oplus B$ is canonically isomorphic to $A \times B$. (Supposing, for example, that $A \times B$ exists, we consider the morphisms $A \longrightarrow A \times B$ and $B \longrightarrow A \times B$ whose components are $(i_A, 0)$, respectively, $(0, i_B)$, we check that we obtain thereby a representation of $A \times B$ as a direct sum of A

[i]Translator's note: Many thanks to George Janelidze for pointing out that Grothendieck's original definition, which assumed only that ϕ and ψ are natural transformations ("homomorphismes de foncteurs"), is insufficient. In fact, he observed that, according to this definition, any two pointed categories would be equivalent.

and B.) Finally, we assume the existence of an object A such that $i_A = 0$; we call it a *zero object* of $\mathbf{C}$. It comes to the same thing to say that $\mathrm{Hom}(A, A)$ is reduced to 0, or that for any $B \in \mathbf{C}$, $\mathrm{Hom}(A, B)$ (or $\mathrm{Hom}(B, A)$) is reduced to 0. If A and A' are zero objects, there exists an unique isomorphism of A to A' (that is, the unique zero element of $\mathrm{Hom}(A, A')$!). We will identify all zero objects of $\mathbf{C}$ to a single one, denoted 0 by abuse of notation.

The dual category of an additive category is still additive.

Let $\mathbf{C}$ be an additive category and $u : A \longrightarrow B$ a morphism in $\mathbf{C}$. For u to be injective (respectively, surjective) it is necessary and sufficient that there not exist a non-zero morphism whose left, respectively, right, composite with u is 0. We call a *generalized kernel* of u any monomorphism $i : A' \longrightarrow A$ such that morphisms from $C \longrightarrow A$ which are right zero divisors of u are exactly the ones that factor through $C \longrightarrow A' \xrightarrow{i} A$. Such a monomorphism is defined up to equivalence (cf. Section 1), so among the generalized kernels of u, if any, there is exactly one that is a subobject of A. We call it *the kernel of* u and denote it by $\mathrm{Ker}\, u$. Dually we define the *cokernel of* u (which is a quotient object of B, if it exists), denoted $\mathrm{Coker}\, u$. We call *image* (respectively, *coimage*) of the morphism u the kernel of its cokernel (respectively, the cokernel of its kernel) if it exists. It is thus a subobject of B (a quotient object of A).[1″] We denote them as $\mathrm{Im}\, u$ and $\mathrm{Coim}\, u$. If u has an image and a coimage, there exists a unique morphism $\overline{u} : \mathrm{Coim}\, u \longrightarrow \mathrm{Im}\, u$ such that u is the composite $A \longrightarrow \mathrm{Coim}\, u \xrightarrow{\overline{u}} \mathrm{Im}\, u \longrightarrow B$, the extreme morphisms being the canonical ones.

A functor F from one additive category $\mathbf{C}$ to another additive category $\mathbf{C}'$ is called an *additive functor* if for morphisms $u, v : A \longrightarrow B$ in $\mathbf{C}$, we have that $F(u+v) = F(u) + F(v)$. The definition for multifunctors is analogous. The composite of additive functors is additive. If F is an additive functor, F transforms a finite direct sum of objects A_i into the direct sum of $F(A_i)$.

1.4 Abelian categories

We call an *abelian category* an additive category $\mathbf{C}$ that satisfies the following two additional conditions (which are self-dual):

AB 1). *Any morphism admits a kernel and a cokernel* (cf. 1.3).

AB 2). *Let u be a morphism in $\mathbf{C}$. Then the canonical morphism $\overline{u} : \mathrm{Coim}\, u \longrightarrow \mathrm{Im}\, u$* (cf. 1.3) *is an isomorphism.*

In particular, it follows that *a bijection is an isomorphism.* Note that there are numerous additive categories that satisfy AB 1) and for which the morphism $\overline{u} : \mathrm{Coim}\, u \longrightarrow \mathrm{Im}\, u$

[1″] A more natural definition of the image of u would be to take the smallest subobject B' of B (if one exists), such that u comes from a morphism of A to B'. This definition is equivalent to the one given in the text only in the case where $\mathbf{C}$ is an *abelian category* (cf. 1.4).

is bijective without being an isomorphism. This is true, for example, for the additive category of separated topological modules over some topological ring, taking as morphisms the continuous homomorphisms as well as for the category of *filtered* abelian groups. A less obvious example: the additive category of holomorphic fibered spaces with vector fibers over a holomorphic variety of complex dimension 1. These are some non-abelian additive categories.

If $\mathbf{C}$ is an abelian category, then the entire usual formalism of diagrams of homomorphisms between abelian groups can be carried over if we replace homomorphisms by morphisms in $\mathbf{C}$, insofar as we are looking at properties of finite type, i.e. not involving infinite direct sums or products (for which special precautions must be taken—see 5). We content ourselves here with indicating a few particularly important facts, referring the reader to [3] for additional details.

In what follows we restrict ourselves to a fixed abelian category $\mathbf{C}$. Let $A \in \mathbf{C}$. To any subobject of A there corresponds the cokernel of its inclusion (which is thereby a quotient of A), and to each quotient object of A there corresponds the kernel of its projection (which is thereby a subobject of A). We thus obtain *one-one correspondence between the class of subjects of A and the class of quotient objects of A*. This correspondence is an *anti-isomorphism* between natural order relations. Moreover, the subobjects of A form a *lattice* (therefore so do the quotient objects): if P and Q are subobjects of A, their sup is the image of the direct sum $P \oplus Q$ under the morphism whose components are the canonical injections of P and Q into A, and their inf is the kernel of the morphism of A to the product $(A/P) \times (A/Q)$, whose components are the canonical surjections to A/P and A/Q. (In accordance with usage, we indicate by A/P the quotient of A corresponding to the cokernel of the inclusion of P into A; it seems natural to use a dual notation such as $A \backslash R$ for the subobject of A that corresponds to the quotient object R. There are dual interpretations for the inf and sup of a pair of quotients objects of A.)

Let $u : A \longrightarrow B$ be a morphism. If A' is a subobject of A, we define the image of A' under u, denoted $u(A')$, as $\operatorname{Im} ui$, where i is the canonical injection $A' \longrightarrow A$. Dually, we define the inverse image $u^{-1}(B')$ of a quotient B' of B; it is a quotient of A. If B' is now a subobject of B, we define the inverse image of B' under u, denoted $u^{-1}(B')$, as the kernel of ju, where j is the canonical surjection $B \longrightarrow B/B'$. We define dually, the direct image $u(A')$ of a quotient A' of A; it is a quotient of B. We show all the usual formal properties for these notions.

Recall that a pair $A \xrightarrow{u} B \xrightarrow{v} C$ of two consecutive morphisms is said to be *exact* if $\operatorname{Ker} v = \operatorname{Im} u$; more generally, we can define the notion of an *exact sequence* of morphisms. For a sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ to be exact it is necessary and sufficient that for $X \in \mathbf{C}$, the following sequence of homomorphisms of abelian groups be exact:

$$0 \longrightarrow \operatorname{Hom}(X, A) \longrightarrow \operatorname{Hom}(X, B) \longrightarrow \operatorname{Hom}(X, C)$$

There is a dual criterion for $A \longrightarrow B \longrightarrow C \longrightarrow 0$ to be exact. Finally, a necessary and sufficient condition that the sequence $0 \longrightarrow A' \xrightarrow{u} A \xrightarrow{v} A'' \longrightarrow 0$ be exact is that u is a monomorphism and that v is a generalized cokernel of u.

Let F be a covariant functor of one abelian category $\mathbf{C}$ to another $\mathbf{C}'$. Following the terminology introduced in [6], we say that F is a *half exact* functor (respectively, *left exact*, respectively, *right exact*) if for any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $\mathbf{C}$, the corresponding sequence of morphisms $0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'') \longrightarrow 0$ is exact at $F(A)$ (respectively, exact at $F(A)$ and $F(A')$, respectively, exact at $F(A)$ and $F(A'')$). F is called an *exact functor* if it is both left exact and right exact, i.e. transforms an exact sequence of the preceding type into an exact sequence; then F transforms *any* exact sequence into an exact sequence. If F is left exact, F transforms an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ into an exact sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$. There is a dual statement for right exact functors. If F is a contravariant functor, we say that F is half exact (respectively, F is left exact, etc.) if F has the corresponding property, as a covariant functor $\mathbf{C}^o$ to $\mathbf{C}'$. The composite of left exact, respectively, right exact, covariant functors is of the same type. We refer back to [6] for further trivialities and for the study of exactness properties of multifunctors. As a significant example, we note that $\mathrm{Hom}(A, B)$ is an additive bifunctor on $\mathbf{C}^o \times \mathbf{C}$, with values in the abelian category of abelian groups, contravariant in A and covariant in B, and left exact with the respect to each argument (that is, in the terminology of [6], a left exact bifunctor).

1.5 Infinite sums and products

In some constructions we will require the existence and certain properties of both infinite direct sums and infinite direct products. Here, in order of increasing strength, are the most commonly used axioms

AB 3). *for any family, $(A_i)_{i\in I}$ of objects of $\mathbf{C}$, the direct sum of the A_i exists (cf. 1).*

This axiom implies that for any family of subobjects (A_i) of an $A \in \mathbf{C}$ the sup of the A_i exists. It suffices to take the image of the direct sum $\bigoplus A_i$ under the morphism whose components are the canonical injections $A_i \longrightarrow A$. We have seen that the direct sum of any family of surjective morphisms is surjective, (no. 1); in fact, we even see that the functor $(A_i)_{i\in I} \mapsto \bigoplus_{i\in I} A_i$, defined over the "product category", $\mathbf{C}^I$ with values in $\mathbf{C}$, is *right exact*. It is even exact if I is finite, but not necessarily if I is infinite, for the direct sum of an infinite family of monomorphisms is not necessarily a monomorphism, as we have noted in 1.1 for the dual situation. Consequently we introduce the following axiom

AB 4). *Axiom* AB 3) *is satisfied and a direct sum of a family of monomorphisms is a monomorphism.*

The following axiom is strictly stronger than AB 4).

AB 5). *Axiom* AB 3) *is satisfied, and if* $(A_i)_{i\in I}$ *is an increasing directed family of subobjects of* $A \in \mathbf{C}$*, and* B *is any subobject of* A*, we have* $\left(\sum_{i\in I} A_i\right) \cap B = \sum_{i\in I}(A_i \cap B)$*.*

(In accordance with normal usage we have denoted by $\sum A_i$ the sup of the A_i, and by $P \cap Q$ the inf of the subobjects P and Q of A.) AB 5) can also be expressed thus: AB 3) *is satisfied, and if* $A \in \mathbf{C}$ *is the sup of an increasing directed family of subobjects* A_i*, and if for any* $i \in I$ *we are given a morphism* $u_i : A_i \longrightarrow B$ *such that when* $A_i \subseteq A_j$*,* $u_i = u_j|A_i$*, then there is a morphism* u *(obviously unique) from* $A \longrightarrow B$ *such that* $u_i = u|A_i$*.* We mention the following axiom that strengthens AB 5), which we will not require in this memoir:

AB 6). *Axiom* AB 3) *holds and for any* $A \in \mathbf{C}$ *and any family* $(B^j)_{j\in J}$ *of increasing directed families of* $B^i = (B^j_i)_{i\in I_j}$ *of subobjects* B^j *of* A*, we have:*

$$\bigcap_{j\in J}\left(\sum_{i\in I_j} B^i_j\right) = \sum_{(i_j)\in\prod I_j}\left(\bigcap_{j\in J} B^i_{i_j}\right)$$

(This axiom implicitly assumes the existence of the inf of any family of subobjects of A.)

We leave it to the reader to state the dual axioms AB 3*), AB 4*), AB 5*), AB 6*), pertaining to infinite products, By way of example, let us point out that the category of abelian groups (or more generally the category of modules over a unital ring), satisfies, with respect to direct sums, the strongest axiom AB 6); it also satisfies axioms AB 3*) and AB 4*), but not AB 5*). The situation is reversed for the dual category, which by the Pontrjagin duality is equivalent to the category of compact topological abelian groups. (This shows that AB 5*) is not a consequence of AB 4*) and hence neither is AB 5) a consequence of AB 4). The abelian category of sheaves of abelian groups over a given topological space X satisfies axioms AB 5) and AB 3*), but not AB 4*), for we have already noted that a product of surjective morphisms need not be surjective. We finish by noting that if $\mathbf{C}$ satisfies both AB 5) and AB 5*), then $\mathbf{C}$ is reduced to the zero object (for we then easily see that for $A \in \mathbf{C}$, the canonical morphism $A^{(I)} \longrightarrow A^I$ is an isomorphism and we may verify that that is possible only when A is zero.[j])

The preceding axioms will be particularly useful for the study of inductive and projective limits that we will need to provide usable existence conditions for "injective" and "projective" objects (see (10)). To avoid repetition we will first study a very general and widely used procedure for forming new categories using diagrams.

[j]Translator's note: It is clear from the claim that by $A^{(I)}$ he means the direct sum of an I-fold of copies of A.

1.6 Categories of diagrams and permanence properties

A *diagram scheme* is a triple (I, Φ, d) made up of two sets I and Φ and a function d from Φ to $I \times I$. The elements of I are *vertices*, the elements of Φ are *arrows* of the diagram and if ϕ is an arrow of the diagram, $d(\phi)$ is called its *direction*, characterized as the *source* and *target* of the arrow (these are therefore vertices of the scheme). A *composite arrow* with source i and target j is, by definition, a non-empty finite sequence of arrows of the diagram, the source of the first being i, the target of each being the source of the next and the target of the last one being j. If $\mathbf{C}$ is a category, we call *diagram in* $\mathbf{C}$ *from the scheme* S a function D which associates to each $i \in I$ an object $D(i) \in \mathbf{C}$ and to any arrow $\phi \in \Phi$ with source i and target j, a morphism $D(\phi) : D(i) \longrightarrow D(j)$. The class of such diagrams will be denoted $\mathbf{C}^S$; it will be considered a category, taking as morphisms from D to D' a family of morphisms $v_i : D(i) \longrightarrow D'(i)$ such that for any arrow ϕ with source i and target j the following diagram commutes:

$$\begin{tikzcd} D(i) \arrow[r, "v_i"] \arrow[d, "D(\phi)"'] & D'(i) \arrow[d, "D'(\phi)"] \\ D(j) \arrow[r, "v(j)"'] & D'(j) \end{tikzcd}$$

Morphisms of diagrams compose in the obvious way, and it is trivial to verify the category axioms. If D is a diagram on the scheme S, then for any composite arrow $\phi = (\phi_1, \ldots, \phi_k)$ in S, we define $D(\phi) = D(\phi_k) \cdots D(\phi_1)$; it is a morphism from $D(i) \longrightarrow D(j)$ if i and j are, respectively, the source and target of ϕ. We call D a *commutative diagram* if we have $D(\phi) = D(\phi')$ whenever ϕ and ϕ' are two composite arrows with the same source and same target. More generally, if R is a set consisting of pairs (ϕ, ϕ') of composite arrows having the same source and target, and of composite arrows whose source equals its target, we consider the subcategory $\mathbf{C}^{S,R}$ of $\mathbf{C}^S$ consisting of diagrams satisfying the *commutativity conditions* $D(\phi) = D(\phi')$ for $(\phi, \phi') \in R$ and $D(\phi)$ is the identity morphism of $D(i)$ if $\phi \in R$ has i as its source and target.

We have to consider still other types of commutation for diagrams, whose nature varies according to the category in question. What follows seems to cover the most important cases. For any $(i, j) \in I \times I$ we take a set R_{ij} of formal linear combinations with integer coefficients of composite arrow with source i and target j, and, if $i = j$, of an auxiliary element e_i. Then if D is a diagram with values in an additive category $\mathbf{C}$, then, for any $L \in R_{ij}$, we can define the morphism $D(L) : D(i) \longrightarrow D(j)$, by replacing, in the expression of L, a composite arrow ϕ by $D(\phi)$ and e_i by the identity element of $D(i)$. If we denote by R the union of the R_{ij}, we will say that D is R-commutative if all the $D(L)$, $L \in R$, are 0. We call a *diagram scheme with commutativity conditions* a pair $(S, R) = \Sigma$ consisting of a diagram scheme S and a set R as above. For any additive category $\mathbf{C}$, we can then consider the subcategory $\mathbf{C}^\Sigma$ of $\mathbf{C}^S$ consisting of the R-commutative diagrams.

1.6.1 PROPOSITION. *Let Σ be a diagram scheme with commutativity conditions and $\mathbf{C}$ an additive category. Then the category $\mathbf{C}^\Sigma$ is an additive category and if $\mathbf{C}$ has infinite direct (respectively, infinite direct sums), so does $\mathbf{C}^\Sigma$. Moreover, if $\mathbf{C}$ satisfies any one of the axioms* AB 1)–AB 6) *or the dual axioms* AB 3*)–AB 6*)*, so does $\mathbf{C}^\Sigma$.*

Moreover, if $D, D' \in \mathbf{C}^\Sigma$, and if u is a morphism from D to D', then its kernel (respectively, cokernel, image, coimage) is the diagram formed by the kernels (respectively, ...) of the components u_i, the morphisms in this diagram (corresponding to the arrows of the scheme) being obtained from those of D (respectively, those of D', ...) in the usual way by restriction (respectively, passage to the quotient). We interpret analogously the direct sum or the direct product of a family of diagrams. Subobjects D' of the diagram D are identified as families $(D'(i))$ of subobjects of $D(i)$ such that for any arrow ϕ with source i and target j we have $D(\phi) : D'(i) \hookrightarrow D'(j)$; then $D'(\phi)$ is defined as the morphism $D'(i) \longrightarrow D'(j)$ defined by $D(\phi)$. The quotient objects of D are defined dually.

If S is a diagram scheme, we call the *dual scheme* and denote it by S^o, the scheme with the same vertices and the same sets of arrows as S, but with the source and target of the arrows of S interchanged. If, moreover, we give a set R of commutativity conditions for S, we will keep the same set for S^o. Using this convention, for an additive category $\mathbf{C}$, the dual category of $\mathbf{C}^\Sigma$ can be identified as $(\mathbf{C}^o)^{\Sigma^o}$.

Let $\mathbf{C}$ and $\mathbf{C}'$ be two additive categories and Σ be a diagram scheme with commutativity conditions. For any functor F from $\mathbf{C}$ to $\mathbf{C}'$, we define in the obvious way the functor F^Σ from $\mathbf{C}^\Sigma$ to C'^Σ, called *the canonical extension of F to the diagram.* F^Σ behaves formally like a Functor with respect to the argument F, in particular, a natural transformation $F \longrightarrow F'$ induces a natural transformation $F^\Sigma \longrightarrow F'^\Sigma$. Finally we note that for a composite functor, we have $(GF)^\Sigma = G^\Sigma F^\Sigma$, and the exactness properties of a functor are preserved by extension to a class of diagrams.

1.7 Examples of categories defined by diagram schemes

(a) Take I reduced to a single element and the empty set of arrows. Then the commutativity relations are of the form $n_i e = 0$, and thus can be reduced to a unique relation $ne = 0$. Then $\mathbf{C}^\Sigma$ is the subcategory of $\mathbf{C}$ consisting of objects annihilated by the integer n. If $n = 0$, we recover $\mathbf{C}$.

(b) Take any I, no arrows, no commutativity relations. Then $\mathbf{C}^\Sigma$ can be identified with the product category $\mathbf{C}^I$. If we suppose we are given commutativity relations, then we get the kind of product category described in 1.5.

(c) Take I reduced to two elements a and b, with a single arrow with source a and target b; we find the category of morphisms $u : A \longrightarrow B$ between objects of $\mathbf{C}$. The introduction of commutativity relations would leave those (a, b, u) that are annihilated by a given integer.

(d) *Categories of functors.* Let $\mathbf{C}'$ be another category, and suppose that it is small. Then the covariant functors from $\mathbf{C}'$ to $\mathbf{C}$ form a category, taking as morphisms the natural transformations (cf. 1.1). This category can be interpreted as a category $\mathbf{C}^{\Sigma}$, where we take $I = \mathbf{C}'$, the arrows with source A' and target B', being by definition the elements of $\mathrm{Hom}(A', B')$ and the commutativity relations being those that express the two functorial axioms. If $\mathbf{C}'$ is also an additive category, the *additive* functors from $\mathbf{C}'$ to $\mathbf{C}$ can also be interpreted as a category $\mathbf{C}^{\Sigma}$ (we add the necessary commutativity relations).

(e) *Complexes with values in* $\mathbf{C}$. $I = \mathbf{Z}$ (the set of integers), the set of arrows being $(d_n)_{n \in \mathbf{Z}}$ where d_n has source n and target $n+1$, the commutativity relations being $d_{n+1} d_n = 0$. We can also add relations of the form $e_n = 0$ if we want to limit ourselves to complexes of positive degrees or to those of negative degrees. We obtain bicomplexes, etc. analogously.

(f) *The category* $\mathbf{C}^G$*, where* G *is a group.* Let G be a group and $\mathbf{C}$ a category (not necessarily additive). We call an *object with a group* G *of operators* in $\mathbf{C}$, a pair (A, r) consisting of an object $A \in \mathbf{C}$ and a representation r of G into the group of automorphisms of A. If (A', r') is a second such pair, we call a morphism from the first to the second a morphism A to A' which commutes with the operations of G. The class $\mathbf{C}^G$ of objects in $\mathbf{C}$ with group G of operators thus becomes a category. We can interpret it as a class $\mathbf{C}^{\Sigma}$ in which we take for $\Sigma = \Sigma(G)$ the following scheme with relations: the set of vertices is reduced to one element i_G, the set of arrows is G, the commutativity relations are $(s)(t) = (st)$ (where the left hand side denotes a composed arrow), and $(e) = e_{i_0}$ (where e denotes the unit element of G). In particular, if $\mathbf{C}$ is additive, the same is true of $\mathbf{C}^G$; in that case our construction is contained in the one that follows (by considering the algebraic relations in the group G).

(g) *The category* $\mathbf{C}^U$ *where* U *is a unital ring.* we consider the additive category consisting of pairs (A, r) of an object A of $\mathbf{C}$ and a unitary representation of U into the ring $\mathrm{Hom}(A, A)$, the morphisms in this category being defined in the obvious way. It is interpreted as above as a category $\mathbf{C}^{\Sigma(U)}$, where $\Sigma(U)$ is the scheme with relations having a single vertex, with U as a set of arrows, and with commutativity relations that we omit.

(h) *Inductive systems and projective systems.* We take as a set of vertices a *preordered* set I, with arrows being pairs (i, j) of vertices with $i \leq j$, the source and target of

(i,j) being i and j, respectively. The commutativity relations are $(i,j)(j,k) = (i,k)$ and $(i,i) = e_i$. The corresponding diagrams (for a give category $\mathbf{C}$, not necessarily additive) are known as *inductive systems* over I with values in $\mathbf{C}$. If we change I to the opposite preordered set, or change $\mathbf{C}$ to $\mathbf{C}^o$ we get *projective systems* over I with values in $\mathbf{C}$. An important case is the one in which I is the lattice of open sets of a topological space X, ordered by containment: we then obtain the notions of *presheaf* over X with values in $\mathbf{C}$.

1.8 Inductive and projective limits

We will discuss only inductive limits, since the notion of projective limit is dual. Let $\mathbf{C}$ be a category, I be a preordered set and $\mathbf{A} = (A_i, u_{i,j})$, be an inductive system over I with values in $\mathbf{C}$ (u_{ij} is a morphism $A_j \longrightarrow A_i$, defined for $i \geq j$). We call a (generalized) *inductive limit* of $\mathbf{A}$ a system consisting of $A \in \mathbf{C}$ and a family (u_i) of morphisms $u_i : A_i \longrightarrow A$, satisfying the following conditions: (a) for $i \leq j$, we have $u_i = u_j u_{ji}$; (b) for every $B \in \mathbf{C}$ and every family (v_i) of morphisms $v_i : A_i \longrightarrow B$, such that for any pair $i \leq j$, the relation $v_i = v_j u_{ji}$ holds, we can find a unique morphism $v : A \longrightarrow B$ such that $v_i = v u_i$ for all $i \in I$. If $(A, (u_i))$ is an inductive limit of $\mathbf{A} = (A_i, u_{ij})$, and if $(B, (v_i))$ is an inductive limit of a second inductive system, $\mathbf{B} = (B_i, v_{ij})$ and finally if $\mathbf{w} = (w_i)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$, then there exists a unique morphism $w : A \longrightarrow B$ such that for all $i \in I$, $w u_i = v_i w_i$. In particular, two inductive limits of the same inductive system are canonically isomorphic (in an obvious way), so it is natural to choose, for every inductive system that admits an inductive limit, a fixed inductive limit (for example, using Hilbert's τ symbol) which we will denote by $\varinjlim \mathbf{A}$ or $\varinjlim_{i \in I} A_i$ and which we will call *the* inductive limit of the given inductive system. If I and $\mathbf{C}$ are such that $\varinjlim \mathbf{A}$ exists for *every* system $\mathbf{A}$ over I with values in $\mathbf{C}$, it follows from the preceding that $\varinjlim \mathbf{A}$ is a *covariant functor* defined over the category of inductive systems on I with values in $\mathbf{C}$.

1.8.1 PROPOSITION. *Let $\mathbf{C}$ be an abelian category satisfying Axiom* AB 3) *(existence of arbitrary direct sums) and let I be an increasing filtered preordered set. Then for every inductive system $\mathbf{A}$ over I with values in $\mathbf{C}$, the $\varinjlim \mathbf{A}$ exists, and it is a right exact additive functor on $\mathbf{A}$. If $\mathbf{C}$ satisfies Axiom* AB 5) *(cf. 1.5), this functor is even exact, and then the kernel of the canonical morphism $u_i : A_i \longrightarrow \varinjlim \mathbf{A}$ is the sup of the kernels of the morphisms $u_{ji} : A_i \longrightarrow A_j$ for $j \geq i$ (in particular, if the u_{ji} are injective, so are the u_i).*

To construct an inductive limit of $\mathbf{A} = (A_i, u_{ij})$ we consider[k] $S = \bigoplus_{i \in I} A_i$ and for every

[k]Translator's note: Grothendieck's argument becomes more or less incoherent at this point and we have substituted a modern argument here.

pair $i \leq j$, $T = \bigoplus_{i \leq j} A_i$. If $v_i : A_i \longrightarrow S$ and $w_{ij} : A_i \longrightarrow T$ are the inclusions into those coproducts, there are two maps $d, e : T \longrightarrow S$ defined as the unique maps for which $dw_{ij} = v_i$ and $ew_i = v_j u_{ij}$s, for all $i \leq j$. Then $\varinjlim \mathbf{A}$ is the coequalizer of d and e. We leave the rest of the proof this proposition to the reader.

1.9 Generators and cogenerators

Let $\mathbf{C}$ be a category, and let $(U_i)_{i \in I}$ be a family of objects of $\mathbf{C}$. We say that it is a *family of generators* of $\mathbf{C}$ if for any object $A \in \mathbf{C}$ and any subobject $B \neq A$, we can find an $i \in I$ and a morphism $u : U_i \longrightarrow A$ which does not come from a morphism $U_i \longrightarrow B$. Then for any $A \in \mathbf{C}$, the subobjects of A form a *set*: in effect, a subobject B of A is completely determined by the set of morphisms of objects $U_i \longrightarrow A$ that factor through B. We say that an object $U \in \mathbf{C}$ is a *generator* of $\mathbf{C}$ if the family $\{U\}$ is a family of generators.

1.9.1 PROPOSITION. *Suppose that* $\mathbf{C}$ *is an abelian category satisfying Axiom* AB 3) *(existence of infinite direct sums). Let* $(U_i)_{i \in I}$ *be a family of objects of* $\mathbf{C}$ *and* $U = \bigoplus U_i$ *its direct sum. The following conditions are equivalent:*

(a) $(U_i)_{i \in I}$ *is a family of generators of* $\mathbf{C}$.

(b) U *is a generator of* $\mathbf{C}$.

(c) *Any* $A \in \mathbf{C}$ *is isomorphic to a quotient of a direct sum* $U^{(I)}$ *of objects that are all identical to* U.

The equivalence of (a) and (b) is an almost immediate consequence of the definition. (b) implies (c), for it is sufficient to take for I the set $\mathrm{Hom}(U, A)$ and to consider the morphism from $U^{(I)}$ to A whose component corresponding to $u \in I$ is u itself: the image B of this morphism is A since otherwise there would exist a $u \in \mathrm{Hom}(U, A) = I$ that does not factor through B, which would be absurd. Thus A is isomorphic to a quotient of $U^{(I)}$. (c) implies (b), for it is immediate that if A is a quotient of $U^{(I)}$, then for any subobject B of A, distinct from A there exists $i \in I$ such that the canonical image in A of the ith factor of $U^{(I)}$ is not contained in B, whence a morphism from U to A that does not factor through B (it can be noted that the additive structure of $\mathbf{C}$ has not been used here).

1.9.2 EXAMPLES. If $\mathbf{C}$ is the abelian category of unitary left modules on a unital ring U, then U (considered as a left module over itself) is a generator. If $\mathbf{C}$ is the category of sheaves of abelian groups over a fixed topological space X, and if for any open $U \subseteq X$, we denote by $\mathcal{Z}_{\mathcal{U}}$ the sheaf on X which is 0 over $\complement U$ and the constant sheaf of integers over U, the family of $\mathcal{Z}_{\mathcal{U}}$ forms a system of generators of $\mathbf{C}$. This example can be immediately generalized to the case in which there is a sheaf $\mathbf{O}$ of rings given over X, and in which we consider the category of sheaves of $\mathbf{O}$-modules over X. There are other examples in the following proposition:

1.9.3 Proposition. *Let Σ be a diagram scheme with commutativity relations (cf. Section 1.6), and let $\mathbf{C}$ be an abelian category and $(U_i)_{i\in I}$ a family of generators of $\mathbf{C}$. Assume that for any arrow of Σ, the source and target of the arrow are distinct, and that the commutativity relations do not involve any identity morphism e_s (where s is a vertex of Σ).*[2] *Then for any $A \in \mathbf{C}$ and any vertex s of the scheme, the diagram $\mathcal{E}_s(A)$ whose value is A at the vertex s and 0 at all other vertices, and whose value at each arrow is 0, belongs to $\mathbf{C}^{\Sigma}$. In addition, the family of $\mathcal{E}_s(U_i)$, (where s and i are variables) is a system of generators for $\mathbf{C}$.*

The proof is immediate; it suffices to note, for the last assertion, that the morphisms of $\mathcal{E}_s(A)$ in a diagram D can be identified with the morphisms of $A \in D(s)$.

We leave it to the reader to develop the dual notion of *family of cogenerators* of an abelian category. We can show that if $\mathbf{C}$ is an abelian category that satisfies Axiom AB 5) (cf. 1.5), then the existence of a generator implies the existence of a cogenerator. (We will not make use of this result.) Thus the category of unitary left modules over a unital ring U always admits a cogenerator: if for example, $U = \mathbf{Z}$, we can take as a cogenerator the group of rational numbers (or the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$).

1.10 Injective and projective objects

Recall that an object M of an abelian category $\mathbf{C}$ is said to be *injective* if the functor $A \mapsto \mathrm{Hom}(A, M)$ (which in any case is left exact) is exact, i.e. if for any morphism $u : B \longrightarrow M$ of a subobject B of an $A \in \mathbf{C}$, there is a morphism of $A \longrightarrow M$ that extends it. A morphism $A \longrightarrow M$ is called an *injective effacement* of A if it is a monomorphism, and if for any monomorphism $B \longrightarrow C$ and any morphism $B \longrightarrow A$, we can find a morphism $C \longrightarrow M$ making the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

commute. Thus, for the identity arrow of M to be an injective effacement, it is necessary and sufficient that M be injective. Any monomorphism into an injective object is an injective effacement.

[2]The category $\mathbf{C}^{\Sigma}$ of inductive systems in $\mathbf{C}$ constructed over a set of *ordered* indices I is relevant in this case. Indeed it suffices, in example 1.7.h to consider the arrows (i, j) for $i < j$, and the commutativity relations $(i, j)(j, k) = (i, k)$, where the e_s play no role.

1.10.1 THEOREM. *If* $\mathbf{C}$ *satisfies Axiom* AB 5) *(cf. 1.5) and admits a generator (cf. 1.9), then any* $A \in \mathbf{C}$ *has a monomorphism into an injective object.*

We will even construct a functor $M : A \mapsto M(A)$ (non-additive in general!) from $\mathbf{C}$ into $\mathbf{C}$ and a natural transformation f of the identity functor into M such that for any $A \in \mathbf{C}$, $M(A)$ is injective and $f(A)$ is a monomorphism of A into $M(A)$. Since the proof is essentially known, we will sketch only the main points.

LEMMA 1. *If* $\mathbf{C}$ *satisfies Axiom* AB 5)*, then the object* $M \in \mathbf{C}$ *is injective if and only if for any subobject* V *of the generator* U*, any morphism* $V \longrightarrow M$ *can be extended to a morphism* $U \longrightarrow M$*.*

It suffices to prove that the condition is sufficient. Thus let u be a morphism from a subobject B of $A \in \mathbf{C}$ to M. We show that there is a morphism of A to M that extends u. We consider the set P of extensions of u to subobjects of A that contain B (it is certainly a set, because by virtue of the existence of a generator, the subobjects of A form a set). We order it by the extension relation. By virtue of the second formulation of Axiom AB 5) (cf. 1.5), this set is inductive. It therefore admits a maximal element; we are thus reduced to the case that u is itself maximal, and to showing that then $B = A$. We prove then that if $B \neq A$, there is an extension of u to a $B' \neq B$. In fact, let $j : U \longrightarrow A$ be a morphism such that $j(U) \not\subseteq B$; set $B' = j(U) + B$ (therefore $B' \neq B$). Let $V = j^{-1}(B)$ be the inverse image of B under j, let $j' : V \longrightarrow B$ be the morphism induced by j, consider the sequence of morphisms $V \xrightarrow{\phi'} U \times B \xrightarrow{\phi} B' \longrightarrow 0$, where the morphism ϕ' has as components the inclusion function of $V \hookrightarrow U$ and $-j'$, and ϕ has as components j and the inclusion function $B \hookrightarrow B'$. We can see immediately that this sequence is exact, so to define a morphism $v : B' \longrightarrow M$, it suffices to define a morphism $w : U \times B \longrightarrow M$ such that $w\phi' = 0$. Now let k be an extension to U of $uj' : V \longrightarrow M$. We take for w the morphism from $U \times V$ to M whose components are k and u. We show immediately that $w\phi' = 0$ and that the morphism $v : B' \longrightarrow M$ induced by w extends u, which completes the proof of Lemma 1. ■

Let $A \in \mathbf{C}$ and let $I(A)$ be the set of all the morphisms u_i from subobjects $V_i(U)$ to A. Consider the morphism $\bigoplus V_i \longrightarrow A \times U^{(I(A))}$ whose restriction to V_i has as components $-u_i : V_i \longrightarrow A$ and the canonical injection of V_i into the ith factor of the direct sum $U^{(I(A))}$. Let $M_1(A)$ be the cokernel of the desired morphism, $f(A) : A \longrightarrow M_1(A)$ be the morphism induced by the canonical epimorphism of $A \times U(I(A))$ over its quotient. Then $f(A)$ is a monomorphism (easily proved thanks to the fact that the canonical morphism $\bigoplus V_i \longrightarrow U^{(I(A))}$ is a monomorphism by AB 4)) and, in addition, any morphism $u_i : V_i \longrightarrow A$ "can be extended" to a morphism $U \longrightarrow M_1(A)$ (in other words, the morphism induced on the ith factor of $U^{(I(A))}$ by the canonical epimorphism of $A \times U^{(I(A))}$ onto its quotient $M_1(A)$). We define by transfinite induction, for any ordinal number i the object $M_i(A)$, and for two ordinal numbers $i \leq j$, an *injective* morphism $M_i(A) \longrightarrow M_j(A)$, such that the

$M_i(A)$, for $i < i_0$ (i_0 being a fixed ordinal number) form an inductive system. For $i = 0$, we will take $M_0(A) = A$; for $i = 1$, $M_1(A)$ and $M_0(A) \longrightarrow M_1(A)$ are already defined. If the construction has been carried out for the ordinals less than i, and if i has the form $j+1$, we set $M_i(A) = M_1(M_j(A))$ and the morphism $M_j(A) \longrightarrow M_{j+1}(A)$ will be $f(M_j(A))$ (which defines at the same time the morphisms $M_k(A) \longrightarrow M_i(A)$ for $k \leq i$). If i is a limit ordinal, we will set $M_i(A) = \lim_{j<i} M_j(A)$, and we will take as morphisms $M_j(A) \longrightarrow M_i(A)$, for $j < i$, the canonical morphisms, which are certainly injective (Proposition 1.8). Now let k be the smallest ordinal whose cardinality is larger than that of the set of subobjects of U (we take $M(A) = M_k(A)$); everything comes down to proving that M_k is injective, i.e. satisfies the condition of Lemma 1. With the notation of this lemma, we prove that $v(V)$ is contained in one of the M_i with $i < k$ (which will complete the proof). In fact, from $M_k = \sup M_i$ we get $V = \sup_{i<k} v^{-1}(M_i)$ (by virtue of AB 5)). Thus since the set of subobjects of V has cardinality less than k, and since any set of ordinal numbers less than k and having k as its limit must itself have cardinality k (since k is not a limit ordinal), it follows that $v^{-1}(M_i)$ stays constant starting from some $i_0 < k$, whence $V = v^{-1}(M_{i_0})$, which completes the proof.

Remark 1. Variant of Theorem 1.10.1: If $\mathbf{C}$ satisfies axioms AB 3), AB 4), and AB 3*) and admits a cogenerator T, then any $A \in \mathbf{C}$ admits an injective effacement. We will not have to use this result.

Remark 2. The fact that $M(A)$ is functorial in A may be convenient, for example, to prove that any $A \in \mathbf{C}^G$ (i.e. an object $A \in \mathbf{C}$ with a group G of operators—cf. 1.7, Example f) which is injective in $\mathbf{C}^G$ is also injective in $\mathbf{C}$.

Remark 3. In many cases, the existence of a monomorphism of A into an injective object can be seen directly in a simpler way. Theorem 1 has the advantage of applying to many different cases. Moreover, the conditions of the theorem are stable under passage to certain categories of diagrams (cf. Propositions 1.6.1 and 1.9.2), in which the existence of sufficiently many injective is not always visible to the naked eye.

Remark 4. We leave it to the reader to provide the dual statements relative to the projective objects and projective effacements.

1.11 Quotient categories

Although they will not be used in the remainder of this work, the considerations of this section, which systematize and make more flexible, the "language modulo $\mathbf{C}$" of Serre, [17], are convenient in various applications.

Let $\mathbf{C}$ be a category. We call a **subcategory** of $\mathbf{C}$ a category $\mathbf{C}'$ whose objects are objects of $\mathbf{C}$, such that for $A, B \in \mathbf{C}'$, the set $\mathrm{Hom}_{\mathbf{C}'}(A, B)$ of morphisms from A to B

in $\mathbf{C}'$ is a subset of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ of morphisms from A to B in $\mathbf{C}$, the composition of morphisms in $\mathbf{C}'$ being induced by the composition of morphisms in $\mathbf{C}$, and the identity morphisms in $\mathbf{C}'$ being the identity morphisms in $\mathbf{C}$. The last two conditions mean that the function that assigns to each object or morphism of $\mathbf{C}'$ the same object or morphism of $\mathbf{C}$ is a covariant functor from $\mathbf{C}'$ to $\mathbf{C}$ (called the *canonical injection from* $\mathbf{C}'$ *to* $\mathbf{C}$). If $\mathbf{C}, \mathbf{C}'$ are additive categories, $\mathbf{C}'$ is called an *additive subcategory* if, in addition to the preceding conditions, the groups $\mathrm{Hom}_{\mathbf{C}'}(A, B)$ are subgroups of $\mathrm{Hom}_{\mathbf{C}}(A, B)$. Suppose that $\mathbf{C}$ is an abelian category. Then $\mathbf{C}'$ is called a *complete*[l]. *It is actually a full abelian subcategory.* if (i) *for* $A, B \in \mathbf{C}'$, *we have* $\mathrm{Hom}_{\mathbf{C}'}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$*; (ii) if in an exact sequence* $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$, *the four extreme terms belong to* $\mathbf{C}'$, *so does the middle term* C. In accordance with (i) the subcategory $\mathbf{C}'$ is completely determined by its objects. (ii) is equivalent to saying the for any morphism $P \longrightarrow Q$ with $P, Q \in \mathbf{C}'$, the kernel and the cokernel belong to $\mathbf{C}'$ and for any exact sequence $0 \longrightarrow R' \longrightarrow R \longrightarrow R'' \longrightarrow 0$, whenever $R', R'' \in \mathbf{C}'$, then $R \in \mathbf{C}'$. We see immediately that then $\mathbf{C}'$ is itself an abelian category and that for a morphism $u : A \longrightarrow B$ in $\mathbf{C}'$, the kernel, cokernel, (and thus the image, and coimage) are identical to the corresponding constructions in $\mathbf{C}$.

The subcategory $\mathbf{C}'$ of $\mathbf{C}$ is called *thick* if it satisfies Condition (i) above and the following strengthening of Condition (ii): (iii) *If in an exact sequence* $A \longrightarrow B \longrightarrow C$, *the outer terms* A *and* C *belong to* $\mathbf{C}'$, *so does* B. If $\mathbf{C}$ is the abelian category of abelian groups, we find the notion of "class of abelian groups" of [17].[m] We see how in [17], (iii) is equivalent to the conjunction of the following three conditions: any zero object belongs to $\mathbf{C}'$; any object that is isomorphic to a subobject or quotient object of $\mathbf{C}'$, belongs to $\mathbf{C}'$; any extension of an object of $\mathbf{C}'$ by an object of $\mathbf{C}'$ belongs to $\mathbf{C}'$.

Let $\mathbf{C}$ be an abelian category and $\mathbf{C}'$ a Serre subcategory. We will define a new abelian category, denoted $\mathbf{C}/\mathbf{C}'$ and called the *quotient category* of $\mathbf{C}$ by $\mathbf{C}'$. The objects of $\mathbf{C}/\mathbf{C}'$ are, by definition, the objects of $\mathbf{C}$. We will define the morphisms in $\mathbf{C}/\mathbf{C}'$ from A to B, called "morphisms mod $\mathbf{C}'$ from A to B". We say that a subobject A' of A is equal mod $\mathbf{C}'$ or quasi-equal to A if A/A' belongs to $\mathbf{C}'$; then any subobject of A containing A' is also quasi-equal to A; moreover, the inf of two subobjects of A that are quasi-equal to A is also quasi-equal to A. Dually, we introduce the notion of quotient of B quasi-equal to B: such a quotient B/N is quasi-equal to B if $N \in \mathbf{C}'$. A morphism mod $\mathbf{C}'$ from A to B is then defined by a morphism f' from a subobject A' of A, quasi-equal to A, to a quotient B' of B quasi-equal to B, it being understood that a morphism $f'' : A'' \longrightarrow B''$ (satisfying the same conditions) defines the same morphism mod $\mathbf{C}'$ if and only if we can find $A''' \leq (A' \wedge A'')$, $B''' \leq (B' \wedge B'')$, A''' quasi-equal to A, B''' quasi-equal to B such that the morphisms $A''' \longrightarrow B'''$ induced by f' and f'' are the same. This last relation between f' and f'' is certainly an equivalence relation and the preceding definition of morphisms mod

[l]Translator's note: We would not today call such a subcategory complete

[m]Translator's note: This is now called a *Serre subcategory*.

$\mathbf{C}'$ is therefore coherent. Suppose that for any $A \in \mathbf{C}$, the subobjects of A form a set (which is true for all known categories). Then we can consider the set of morphisms mod $\mathbf{C}'$ from A to B, denoted $\mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, B)$. $\mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, B)$ appears as the inductive limit of the abelian groups $\mathrm{Hom}_{\mathbf{C}}(A', B')$ where A' ranges over the subobjects of A quasi-equal to A, and B' ranges over the quotients of B quasi-equal to B and is consequently an abelian group. We similarly define a pairing $\mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, B) \times \mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(B, C) \longrightarrow \mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, C)$ as follows. Let $u \in \mathrm{Hom}_{\mathbf{C}}(A', B')$ represent $u' \in \mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, B)$ and $v \in \mathrm{Hom}_{\mathbf{C}}(B'', C')$ represent $v' \in \mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(B, C)$. Let Q be the image of the canonical morphism $B'' \longrightarrow B'$ of the subobject B'' of B in the quotient B'; Q is also isomorphic to the coimage of this morphism and is therefore both a subobject of B' and a quotient object of B''. By decreasing, if necessary, the subobject A' of A and the quotient object C' of C, we can assume that u and v come from morphisms (denoted by the same letters), $A' \longrightarrow Q$ and $Q \longrightarrow C'$. We can now take the composite $vu \in \mathrm{Hom}_{\mathbf{C}}(A', C')$ and we verify that the element of $\mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, C)$ that this defines depends only on u' and v'. We denote it $v'u'$. There is no difficulty in proving that the law of composition thus defined is bilinear and associative, and the that the class in $\mathrm{Hom}_{\mathbf{C}/\mathbf{C}'}(A, A)$ of the identity morphism i_A is a universal unit, so $\mathbf{C}/\mathbf{C}'$ is an additive category, and finally that it is even an abelian category. We will not complete the extremely tedious proof. Thus $\mathbf{C}/\mathbf{C}'$ *appears as an abelian category; moreover the identity functor* $F : \mathbf{C} \longrightarrow \mathbf{C}/\mathbf{C}'$ *is exact* (and, in particular, commutes with kernels, cokernels, images, and coimages)*,* $F(A) = 0$ *if and only if* $A \in \mathbf{C}'$*, and any object of* $\mathbf{C}/\mathbf{C}'$ *has the form* $F(A)$ *for some* $A \in \mathbf{C}$*.* These are the facts (which essentially characterize the quotient category) which allow us to safely apply the "mod $\mathbf{C}'$" language, since this language signifies simply that we are in the quotient abelian category. It is particularly convenient to use, when we have a spectral sequence (cf. 2.4) in $\mathbf{C}$, the fact that some terms of the spectral sequence belong to $\mathbf{C}'$: reducing mod $\mathbf{C}'$ (i.e. applying the functor F), we find a spectral sequence in $\mathbf{C}/\mathbf{C}'$ in which the corresponding terms vanish, whence we have exact sequences mod $\mathbf{C}'$, with the help of the usual criteria for obtaining exact sequences from spectral sequences in which certain terms have vanished.

Chapter 2

Homological algebra in abelian categories

2.1 ∂-functors and ∂^*-functors

Let $\mathbf{C}$ be an abelian category, $\mathbf{C}'$ an additive category, and a and b be two integers (which can be equal to $\pm\infty$) such that $a+1<b$. A *covariant δ-functor from* $\mathbf{C}$ *to* $\mathbf{C}'$ *in degrees* $a<i<b$, is a system $T=(T^i)$ of additive covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, $(a<i<b)$, in addition to giving, for any i such that $a<i<b-1$ and for any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, a morphism

$$\partial : T^i(A'') \longrightarrow T^{i+1}(A')$$

(the "boundary" or "connecting" homomorphism). The following axioms are assumed to be satisfied:

(i) If we have a second exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ and a homomorphism from the first exact sequence to the second, the corresponding diagram

$$\begin{tikzcd}
T^i(A'') \arrow[r, "\partial"] \arrow[d] & T^{i+1}(A') \arrow[d] \\
T^i(B'') \arrow[r, "\partial"'] & T^{i+1}(B')
\end{tikzcd}$$

commutes.

(ii) For any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, the associated sequence of morphisms

(2.1.1) $$\cdots \longrightarrow T^i(A') \longrightarrow T^i(A) \longrightarrow T^i(A'') \longrightarrow T^{i+1}(A') \longrightarrow \cdots$$

is a complex, i.e. the composite of two consecutive morphisms in this sequence is 0.

There is an analogous definition for a covariant ∂^*-functor, the only difference being that the ∂^* operator decreases the degree by one unit instead of increasing it. There are analogous definitions for contravariant ∂-functors and ∂^*-functors. The T^i are then contravariant additive functors and the boundary operators go from $T^i(A') \longrightarrow T^{i+1}(A'')$ or $T^i(A') \longrightarrow T^{i-1}(A'')$. If we change the sign of the i in T^i, or if we replace $\mathbf{C}'$ by its dual, the ∂-functors become ∂^*-functors. Thus, one can always stick to the study of covariant ∂-functors. Note that if $a = -\infty,\ b = +\infty$, a ∂-functor is a connected sequence of functors as in [6, Chapter III].

Given two ∂-functors T and T' defined in the same degrees, we call a *morphism* (or natural transformation) from $T \longrightarrow T'$ a system $f = (f^i)$ of natural transformations $f^i : T^i \longrightarrow T'^i$ subject to the natural condition of commutativity with ∂: for any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, the diagram

$$\begin{tikzcd}
T^i(A'') \arrow[r, "\partial"] \arrow[d] & T^{i+1}(A') \arrow[d] \\
T'^i(A'') \arrow[r, "\partial"'] & T'^{i+1}(A')
\end{tikzcd}$$

commutes. Morphisms of ∂-functors add and compose in the obvious way.

Assume that $\mathbf{C}'$ is also an abelian category. A ∂-functor is *exact* if for any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $\mathbf{C}$, the corresponding sequence (2.1.1) is exact. We say that a *cohomological functor* (respectively *homological functor*) is an *exact* ∂-functor (respectively *exact* ∂^*-functor) defined for all degrees.

2.2 Universal ∂-functors

Let $T = (T^i)$ for $0 \leq i \leq a$ be a covariant ∂-functor from $\mathbf{C}$ to $\mathbf{C}'$, where $a > 0$. T is called a *universal* ∂-functor if for any ∂-functor $T' = (T'^i)$ defined in the same degrees, any natural transformation $f^0 : T^0 \longrightarrow T'^0$ extends to a unique ∂-functor $f : T \longrightarrow T'$ which reduces to f^0 in degree 0.[n] We use the same definition for contravariant ∂-functors. In the case of ∂^*-functors we have to consider morphisms from $T' \longrightarrow T$ rather than $T \longrightarrow T'$.

By definition, given a covariant functor F from $\mathbf{C}$ to $\mathbf{C}$', and an integer $a > 0$, there can exist, up to unique isomorphism, at most one universal ∂-functor defined in degrees $0 \leq i \leq a$ and reducing to F in degree 0. Its components are then denoted S^iF and called the *right satellite functors* of F. If $i \leq 0$, we also set $S_iF = S^{-i}F$, where the S_iF are *left*

[n]Translator's note: The original is more or less incoherent (for example, uses notation T^{i0}, where we have T^0) and we have changed it to what has to be intended.

satellites of F, defined as S^iF, by considering the universal ∂^*-functors, defined in degrees $0 \leq i \leq a$, such that $T^0 = F$. We can then show immediately that for a given i, if S^iF exists, it is independent of the choice of a.

In all cases I am aware of, the satellite functors of any additive functor F exist. Moreover, if $\mathbf{C}, \mathbf{C}'$ are given, to show that for any additive covariant functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ there is a universal ∂-functor in all degrees and reducing to F in degree 0 (i.e. all the satellites S^iF exist), it clearly suffices to prove that S^1F and $S^{-1}F$ exist, thanks to the equations $S^1(S^iF) = S^{i+1}F$ if $i > 0$ and $S^{-1}(S^iF) = S^{i-1}F$, if $i \leq 0$ (for which the proof follows trivially from the definition). Moreover, the search for S^1F and $S^{-1}F$ are dual problems; these functors exchange if we replace $\mathbf{C}$ and $\mathbf{C}'$ by the dual categories.

An additive functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ is called *effaceable* if for any $A \in \mathbf{C}$ we can find a monomorphism $u : A \longrightarrow M$ such that $F(u) = 0$; if $\mathbf{C}$ is such that any $A \in \mathbf{C}$ admits an injective effacement (cf. Remark 1.10.1) this is equivalent to saying that $F(u) = 0$ for any injective effacement u; if $\mathbf{C}$ is such that any object $A \in \mathbf{C}$ admits a monomorphism into an injective object M (cf. 1.10.1), it is equivalent to saying that $F(M) = 0$ for any injective object M. Dually, F is said to be *coeffaceable* if for any $A \in \mathbf{C}$, we can find an epimorphism $u : P \longrightarrow A$ such that $F(u) = 0$.

2.2.1 Proposition. *Let $\mathbf{C}$ and $\mathbf{C}'$ be two abelian categories and $T = (T^i)$, $0 < i < a$ an exact ∂-functor (covariant or contravariant) from $\mathbf{C}$ to $\mathbf{C}'$ with $a > 0$. If T^i is effaceable for $i > 0$ then T is a universal ∂-functor, and the converse is true if $\mathbf{C}$ is such that any object $A \in \mathbf{C}$ admits an injective effacement (cf. 1.10).*

It suffices for the direct part to prove, for example, that if (T'^0, T'^1) is a ∂-functor defined in degrees 0 and 1 and f^0 is a natural transformation $T^0 \longrightarrow T'^0$ then there exists a unique morphism $f^1 : T^1 \longrightarrow T'^1$ such that $(f^0, f^1) : (T^0, T^1) \longrightarrow (T'^0, T'^1)$ is a natural transformation of ∂-functors (we have chosen to consider case that T is covariant). Let $A \in \mathbf{C}$. We consider an exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow A' \longrightarrow 0$ such that the morphism $T^1(A) \longrightarrow T^1(M)$ is zero. If we have been able to construct f^1 we will have a commutative diagram

$$\begin{tikzcd}
T^0(M) \arrow[r] \arrow[d] & T^0(A') \arrow[r] \arrow[d] & T^1(A) \arrow[r, "0"] \arrow[d] & T^1(M) \\
T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^1(A) &
\end{tikzcd}$$

Since the first line is exact, we conclude that the morphism $T^0(A') \longrightarrow T^1(A)$ is surjective, and consequently the morphism $f^1(A) : T^1(A) \longrightarrow T'^1(A)$ is completely determined by passage to the quotient starting from $f^0(A') : T^0(A') \longrightarrow T'^0(A')$. This proves the uniqueness of $f^1(A)$. Moreover, the preceding diagram without $f^1(A)$, taking into account that the composite of the two morphisms of the second row is 0, allows us to define a morphism

$T^1(A) \longrightarrow T'^1(A)$ determined uniquely by the condition that the diagram remain commutative. Standard reasoning shows that the morphism thus defined does not depend on the particular choice of the exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow A' \longrightarrow 0$, thus showing that the morphism is functorial and "commutes with ∂". That proves the first part of the proposition. The second part is contained in the following existence theorem:

2.2.2 THEOREM. *Let $\mathbf{C}$ be an abelian category such that any object $A \in \mathbf{C}$ admits an injective effacement (cf. 1.10). Then for any covariant additive functor F on $\mathbf{C}$, the satellites S^iF, ($i \geq 0$) exist and are effaceable functors for $i > 0$. For the universal ∂-functor $(S^iF)_{i\geq 0}$ to be exact, it is necessary and sufficient that F satisfy the following condition: F is half exact, and for $P \subseteq Q \subseteq R$ in $\mathbf{C}$, the kernel of $F(Q/P) \longrightarrow F(R/P)$ is contained in the image of $F(Q) \longrightarrow F(Q/P)$ (conditions always satisfied if F is either left or right exact).*[2']

S The proof is essentially contained in [6, Chapter III]. For the first part, it suffices to prove the existence of S^1F. Let $A \in \mathbf{C}$. We consider an exact sequence $0 \longrightarrow A \longrightarrow M \longrightarrow A' \longrightarrow 0$ where the first morphism is an injective effacement of A. Let $S^1F(A) = F(A')/\operatorname{Im}(F(M))$. We see as in [6] that the second term is independent of the particular exact sequence chosen (modulo canonical isomorphisms) and can be considered a functor in A. The definition of the boundary homomorphism, the proof of axioms (1) and (2) of 2.1 and the fact that the ∂-functor obtained (F, S^1F) is universal is also standard. Similarly, we will omit the proof of the exactness criterion. We point out the dual statement. *If in $\mathbf{C}$, every object admits a projective effacement, then the satellites S^iF ($i \leq 0$) exist and are coeffaceable functors; the condition for the ∂-functor $S^iF_{i\leq 0}$ to be exact is the same as in the statement of Theorem 2.2.2.* Consequently, if every object of $\mathbf{C}$ admits both an injective and a projective effacement then every additive covariant functor F admits satellites S^iF for any i; and for the universal ∂-functor S^iF to be exact, it is necessary and sufficient that F satisfy the condition given in the statement of Theorem 2.2.2. If F is a contravariant functor, it is necessary, in the above statements, to exchange the negative and positive indices and replace the exactness condition by a dual condition.

REMARK. We point out another case, very different from the one in Theorem 2.2.2, where we can construct the satellite functors of an arbitrary functor from $\mathbf{C}$ to $\mathbf{C}'$. Suppose that we can find a *set* $\mathbf{C}_0 \subseteq \mathbf{C}$ such that any $A \in \mathbf{C}$ is isomorphic to an object in $\mathbf{C}_0$, and suppose that $\mathbf{C}'$ is an abelian category in which infinite direct sums exist. Then for any additive functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$, the satellites (S^iF), $i \geq 0$ exist. Moreover, if $\mathbf{C}'$ satisfies axiom AB 5) (cf. 1.5) and if F satisfies the condition at the end of Theorem 2.2.2, then the ∂-functor $(S^iF_{i\geq 0})$ is *exact*. Since, in particular, the category of abelian groups satisfies condition AB 5), we can apply the preceding result to the functor $\operatorname{Hom}(A, B)$ with values

[2'] (Note added in proof) This condition is automatically satisfied if every object of $\mathbf{C}$ is isomorphic to a subobject of an injective object, cf. [6, Chapter III].

in the category of abelian groups, and thereby define the functors $\mathrm{Ext}^i(A, B)$ as satellites of $\mathrm{Hom}(A, B)$, considered either as a covariant functor in B or a contravariant functor in A. (But it remains to be proved that two procedures give the same result.) The above-mentioned condition on $\mathbf{C}$ is satisfied for categories whose objects are subject to certain finiteness conditions (and where, in particular, infinite direct sums do not in general exist). Example: the abelian category of those algebraic groups (not necessarily connected) defined over a fixed field K of characteristic 0, which are complete as algebraic varieties and abelian as groups, i.e. the category of algebraic abelian groups defined over K whose connected component at 0 is an "abelian variety" (we are forced to assume that the characteristic is 0, otherwise a bijective homomorphism would not necessarily be an isomorphism). We indicate only that we prove the result stated above by constructing $S^iF(A)$ as an inductive limit of the objects $F(M/A)/\operatorname{Im}(F(M))$ for "all" monomorphisms $A \rightarrowtail M$ in $\mathbf{C}$, preordered by saying $A \rightarrowtail M$ is below $A \rightarrowtail M'$ if we can find a morphism $M \longrightarrow M'$ that restricts to the identity on A.

2.3 Derived functors

Let $\mathbf{C}$ and $\mathbf{C}'$ be abelian categories. The theory of derived functors of an additive functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ is developed as in [6, Chapter V], subject to the assumption that any object of $A \in \mathbf{C}$ is isomorphic to a subobject of an injective object or to a quotient object of a projective object or both. Thus, in order to define *right* derived functors of a *covariant* functor or *left* derived functors of a *contravariant* functor, it is necessary to assume that every object $A \in \mathbf{C}$ is isomorphic to a subobject of an injective object, whence we conclude that every $A \in \mathbf{C}$ admits an injective resolution[3]: $0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$, whence the definition of $R^iF(A) = H^i(F, C)$ (where C denotes the complex of the C^i). If we wish to define the left derived functors of a covariant functor or the right derived functors of a contravariant functor, it is similarly necessary to assume that every object $A \in \mathbf{C}$ is isomorphic to a quotient of a projective object. And finally, to define the derived functors of a mixed functor in several variables, it is necessary to make the appropriate assumption about the domain category of each variable, adapting the exposition of [6] as necessary. In particular, if F is, for example, covariant and if we can form its right derived functors R^iF, then (setting $R^iF = 0$ when $i < 0$) $RF = (R^iF)$ is a cohomological functor (called the *right derived cohomological functor* of F), and we have a canonical morphism of ∂-functors of positive degrees $SF \longrightarrow RF$ (where $SF = (S^iF)$ is the universal ∂-functor of positive degrees, the satellite of F, which exists by virtue of Theorem 2.2.2); this last is an

[3] A *right resolution* $A \longrightarrow C$ is, by definition, a complex C of positive degree equipped with an "augmentation homomorphism" $A \longrightarrow C$ (A being considered as a complex concentrated in degree 0), such that the sequence $0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$ is exact. We call an *injective resolution* of A a resolution C of A such that the C^i are injective objects. *Left* resolutions of A and *projective* resolutions of A are defined dually.

isomorphism if and only if F is left exact. We note that it seems that the consideration of the R^iF is not very interesting unless F is left exact, i.e., when they coincide with the satellite functors. However, the simultaneous definition of the R^iF by injective resolutions is easier to deal with than the recursive definition of the S^iF and, in particular, lends itself better to the construction of the most important spectral sequences (see 2.4).

Let $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}'$ be three abelian categories. Let $T(A, B)$ be an additive bifunctor $\mathbf{C}_1 \times \mathbf{C}_2 \longrightarrow \mathbf{C}'$ which, to fix the ideas, we will assume to be contravariant in A and covariant in B. Suppose that every object of C_2 is isomorphic to a subobject of an injective object. Then we can construct the right partial derived functors of T with respect to the second variable B.

$$R_2^iT(A, B) = H^i(T(A, C(B)))$$

where $C(B)$ is the complex defined by a right resolution of B by injective objects. Of course, the R_2^iT are bifunctors. *We now suppose that for any injective object B in $\mathbf{C}_2$, the functor defined on $\mathbf{C}_1$ by $A \mapsto T(A, B)$ is exact.* We will show that *for any $B \in \mathbf{C}_2$* the sequence $(R_2^iT(A, B))$ can be considered a cohomological functor in A. Let $C(B)$ be the complex defined by a right resolution of B by injective objects. For any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ in $\mathbf{C}_1$, the sequence of homomorphisms of complexes $0 \longrightarrow T(A'', C(B)) \longrightarrow T(A, C(B)) \longrightarrow T(A', C(B)) \longrightarrow 0$ is exact (according to the assumption on T, the terms of $C(B)$ are injective), and it thus defines an exact cohomology sequence, that is,

$$\cdots \longrightarrow R_2^iT(A'', B) \longrightarrow R_2^iT(A, B) \longrightarrow R_2^iT(A', B) \xrightarrow{\partial} R_2^{i+1}T(A'', B) \longrightarrow \cdots$$

We can immediately verify that the morphism ∂ in this sequence does not depend on the specific choice of the complex $C(B)$ and that it commutes with homomorphisms of exact sequences, which shows that, for fixed B, $(R_2^iT(A, B))$ is a cohomological functor in A. Moreover, we can also verify immediately that, for a morphism $B \longrightarrow B'$ in $\mathbf{C}_2$, the corresponding morphisms $R_2^iT(A, B) \longrightarrow R_2^iT(A, B')$ define a morphism of ∂-functors in A. If we assume that in $\mathbf{C}_1$, every object is isomorphic to a quotient of a projective object, and for every projective object $A \in \mathbf{C}_1$, the functor $T(A, B)$ is exact in B (in which case, we say, in accordance with the terminology of [6], that T is "right balanced"), then the $R_2^iT(A, B)$ are also the right derived functors $R^iT(A, B)$ of T, and also coincide with the partial derived functors $R_1^iT(A, B)$; in this case, the boundary homomorphisms constructed above are nothing other than the natural boundary morphisms of R^iT and R_1^iT with respect to their first variable.

The preceding considerations were mainly developed for the case of an abelian category $\mathbf{C}$ in which every object is isomorphic to a subobject of an injective object, but in which we do not assume that every object is isomorphic to a quotient of a projective object. We then take $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{C}$ and $\mathbf{C}'$ is the category of abelian groups, $T(A, B) = \mathrm{Hom}(A, B)$; according to the definition of injective object, $\mathrm{Hom}(A, B)$ is exact in A if B is injective.

Denoting by $\mathrm{Ext}^p(A,B)$ the right derived functors with respect to B, we see that the $\mathrm{Ext}^p(A,B)$ form a cohomological functor in B *and* in A. We have $\mathrm{Ext}^0(A,B) = \mathrm{Hom}(A,B)$ since $\mathrm{Hom}(A,B)$ is left exact; it is easy to see that $\mathrm{Ext}^1(A,B)$ can also be interpreted as the group of classes of extensions of A (quotient) by B (subobject), [6, Chapter 14].

An important case in which the abelian category $\mathbf{C}$ contains enough injectives but not enough projectives is the one in which $\mathbf{C}$ is the category of sheaves of modules over a given sheaf of rings on a topological space X. In Chapter IV we will study in greater detail the groups $\mathrm{Ext}^p(A,B)$ in this case.

2.4 Spectral sequences and spectral functors

For the theory of spectral sequences, we refer to [6, Chapters XV and XVII] confining ourselves to specifying our terminology and emphasizing the most useful general cases in which spectral sequences can be written.

Let $\mathbf{C}$ be an abelian category. Let $A \in \mathbf{C}$. A (decreasing) *filtration* on A is a family $(F^n(A))$ $(n \in \mathbf{Z})$ of subobjects of A with $F^n(A) \subseteq F^{n'}(A)$ if $n \geq n'$. A *filtered object* in $\mathbf{C}$ is an object of $\mathbf{C}$ equipped with a filtration. If A and B are two filtered objects of $\mathbf{C}$, a morphism $u : A \longrightarrow B$ is said to be compatible with the filtration if $u(F^n(A)) \subseteq F^n(B)$ for any n. With this notion of morphism, the objects with filtration in $\mathbf{C}$ form an additive category (but not an abelian category, for a bijection in this category is not necessarily an isomorphism!). We call the family of $G^n(A) = F^n(A)/F^{n+1}(A)$ the graded object associated with the filtered object A; we denote it by $G(A)$.[4] $G(A)$ is a covariant functor with respect to the filtered object A. A *spectral sequence* in $\mathbf{C}$ is a system $E = (E_r^{p,q}, E^n)$ consisting of

(a) objects $E_r^{p,q} \in \mathbf{C}$ defined for integers p, q, r with $r \geq 2$ (We can replace 2 by any integer r_0 but in the applications, only the cases $r = 2$ and $r = 1$ seem interesting);

(b) morphisms $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} d_r^{p,q} = 0$;

(c) isomorphisms $\alpha_r^{p,q} : \ker(d_r^{p,q})/\operatorname{Im}(d_r^{p-r,q+r-1}) \longrightarrow E_{r+1}^{p,q}$;

(d) filtered objects $E^n \in \mathbf{C}$, defined for any integer n. We assume that for any fixed pair (p,q), we have $d_r^{p,q} = 0$ and $d_r^{p-r,q+r-1} = 0$ for r sufficiently large, from which we conclude that $E_r^{p,q}$ is independent of r for a sufficiently large r; we denote this object

[4] These definitions are not self-dual. We re-establish the duality by associating with any "decreasing filtration" of A by the $F^n(A)$ the associated "decreasing cofiltration" by $F'_n = A/F_{1-n}(A)$. Then passing from a category to the dual category, these two filtrations become, respectively, a decreasing cofiltration and an associated decreasing filtration. It is still convenient to set $F_n(A) = F^{1-n}(A)$, $F'^n(A) = F'_{1-n}(A)$ (the *increasing* filtration and cofiltration associated with the preceding filtrations), and depending on the case, one of the four will be the most convenient of the associated filtrations to consider. Setting $G^n(A) = \mathrm{Coker}(F^n(A) \longrightarrow F^{n-1}(A))$, $G_n(A) = \ker(F'_n(A) \longrightarrow F'_{n-1}(A))$, we will have $G^n(A) = G_{-n}(A)$. Thus functors G^n and G_n exchange as we pass from $\mathbf{C}$ to the dual category.

by $E_\infty^{p,q}$. We assume, in addition, that for any fixed n, $F^p(E^n) = E^n$ for a sufficiently small p and is 0 for a sufficiently large p.

(e) We assume given isomorphisms $\beta^{p,q} : E_\infty^{p,q} \longrightarrow G^p(E^{p+q})$[o].

The family (E^n) without filtrations is called the *abutment* of the spectral sequence E.

A *morphism u from one spectral sequence* $E = (E_r^{p,q}, E^n)$ to another $E' = (E_r'^{p,q}, E'^n)$ consists of a system of morphisms $u_r^{p,q} : E_r^{p,q} \longrightarrow E_r'^{p,q}$, and $u^n : E^n \longrightarrow E'^n$ compatible with the filtrations, these morphisms being subject to commutativity with morphisms $d_r^{p,q}, \alpha_r^{p,q}$, and $\beta^{p,q}$. The spectral sequences in $\mathbf{C}$ then form an additive category (but not, of course, abelian). We call a *spectral functor* an additive functor defined on an abelian category, with values in a category of spectral sequences[5]. A spectral sequence is said to be a *cohomological spectral sequence* if $E_r^{p,q} = 0$ when $p < 0$ or when $q < 0$. Then we have $E_r^{p,q} = E_\infty^{p,q}$ when $r > \sup(p, q+1)$, $E^n = 0$ for $n < 0$, $F^m(E^n) = 0$ if $m > n$, and $F^m(E^n) = E^n$ if $m \leq 0$.

For example, let K be a *bicomplex*[6] in $\mathbf{C}$, $K = (K^{p,q})$, and assume that for any integer n there are only finitely many pairs (p, q) such that $p+q = n$ and $K^{p,q} \neq 0$. Then we can find two spectral sequences, both convergent to $H(K) = (H^n(K))$ (cohomology of the single complex K^n associated to K, $K^n = \sum_{p+q=n} K^{p,q}$) and whose first terms are, respectively,

$$\mathrm{I}_2^{p,q}(K) = H_{\mathrm{I}}^p H_{\mathrm{II}}^q(K) \qquad \mathrm{II}_2^{p,q}(K) = H_{\mathrm{II}}^p H_{\mathrm{I}}^q(K) \tag{2.4.1}$$

(using the notation of [6, Chapter XV, no. 6]). These spectral sequences are spectral functors in K. Moreover they are *cohomological* spectral sequences if the degrees of K are positive.

Let F be a covariant functor from one abelian category $\mathbf{C}$ to another $\mathbf{C}'$. We assume that in $\mathbf{C}$ every object is isomorphic to a subobject of an injective object. Let $K = (K^n)$ be a complex in $\mathbf{C}$ bounded on the left (i.e. $K^n = 0$ for all sufficiently small n). Then the considerations of [6, Chapter XVII] apply and allow the construction of two spectral sequences, $(\mathrm{I}\,F_r^{p,q}(K), \mathrm{I}\,F^n(K))$ and $(\mathrm{II}\,F_r^{p,q}(K), \mathrm{II}\,F^p(K))$, convergent to the same graded object $\mathcal{R}F(K) = (\mathcal{R}^n F(K))$ (for two appropriate filtrations) and whose initial terms are, respectively,

$$\mathrm{I}\,F_2^{p,q}(K^p) = H(R^q F(K)) \qquad \mathrm{II}\,F_2^{p,q}(K) = R^p F(H^q(K)) \tag{2.4.2}$$

[o]Translator's note: In the original, the domain of $\beta^{p,q}$ is given as E^{pq}, which makes little sense. This is out best guess as to what it should have been, especially since that notation has just been defined and is not subsequently used. In any case this approach to spectral sequences does not appear to have caught on.

[5]It seems that for all the known spectral functors, the abutment is, in fact, a cohomological functor. The relations between boundary homomorphisms and the other constituents of the spectral functor remain to be examined.

[6]Contrary to the terminology introduced in [6], we assume that the two boundary operators d' and d'' of a bicomplex K *commute*, and we therefore take as the *total* boundary operator the morphism d, defined by $dx = d'x + (-1)^p d''x$ for $x \in K^{p,q}$.

(As is customary, a functor applied to a complex K denotes the complex gotten by applying the functor to each term of the complex.) These spectral sequences are functorial in K and the variance of the initial and final terms is evident from their explicit form. We have thus defined two spectral functors on the category of complexes in $\mathbf{C}$ that are bounded on the left; we call them the *right derived spectral functors* of F or the *hyperhomology* right *spectral functors of* F. The functors $\mathcal{R}^n F(K)$ are called the *hyperhomology functors* of F.

We recall the principle of the construction, limiting ourselves to the case that K is of positive degree in order to fix the ideas. Let $L = (L^{p,q})$ be a double complex, of positive degrees, equipped with an augmentation $K \longrightarrow L$ (where K is considered a bicomplex in which the second degree is 0); we assume that for every p, the complex $L^{p,*} = (L^{p,q})_q$ is a *resolution* of K^p and that for any p, q, we have $R^n F(L^{p,q}) = 0$ for $n > 0$. Here are two specific ways, both unique up to homotopy equivalences of bicomplexes, to construct such a bicomplex: (a) we consider K an object of the abelian category $\mathcal{K}$ of complexes in $\mathbf{C}$ of positive degree, and we take for L an *injective* resolution of the object K. We can easily see that the injective objects in $\mathcal{K}$ are the complexes $A = (A^i)$ of positive degree such that every A^i is injective and $H^i(A) = 0$ for $i > 0$, which "decompose" (i.e. the subobjects $Z(A^i)$ of cycles are direct summands of A^i). Moreover, every object of $\mathcal{K}$ embeds into an injective object. (b) We take an "injective resolution" of K in the sense of [6, Chapter XVII] (the quotation marks are necessary because the terminology of [6] is patently ambiguous), i.e., we assume that the $L^{p,q}$ are injective, and that for fixed p, if we take the cycles (respectively the limits, respectively the cohomology) of the $L^{p,*}$ with respect to the first differential operator, we find an injective resolution of the object of cycles (respectively limits, respectively cohomology), of K^p. This having been said, if L is as above, then $H^* F(L)$ is essentially independent of the choice of L, and moreover is identified as $R^*(F \mathrm{o} H^0)(K)$ (where H^0 is considered a left exact covariant functor $\mathcal{K} \longrightarrow \mathbf{C}$, and $F \mathrm{o} H^0$ is the composite functor $\mathcal{K} \longrightarrow \mathbf{C}'$). To see this, it suffices to take an injective resolution L' of K (in the sense of (a)): there is therefore a homomorphism (unique up to homotopy) from the resolution L to the resolution L', whence a homomorphism $F(L) \longrightarrow F(L')$, which induces an *isomorphism* $\mathrm{I}_2^{p,q}(F(L)) \longrightarrow \mathrm{I}_2^{p,q}(F(L'))$ (the two members can be identified with $H^p(R^1 F(K)))$ thus an *isomorphism* of $HF(L)$ onto $HF(L')$. This last becomes explicit as a result of second spectral sequence of the bicomplex $F(L')$: we have $H_1^q(F(L')) = 0$ for $q > 0$ as we see immediately. We therefore find $H^n F(L') = H^n(H^0(L'^{*,n}))$; the second member is, by definition, $R^n(F \mathrm{o} H^0)(A)$. Whence the definition and the general method of calculating the *hyperhomology* $\mathcal{R}F(K) = \mathcal{R}^*(FH^0)(K)$ of the functor F with respect to the complex K and of the first spectral sequence, whose initial term is $\mathrm{I}_2^{p,q} F(K) = H^p(R^q F(K))$ which converges to it. If we now take for L an "injective resolution" of K as in (b), then the second spectral sequence of the bicomplex $F(L)$ is essentially independent of L (since L is unique up to homotopy equivalence); it converges to $\mathcal{R}F(K)$ and its initial term is the one given in 2.4.2. It would suffice, moreover, to have $R^n F(L^{p,q}) = 0$ for $n > 0$ (instead of injective $L^{p,q}$) in the statement of the conditions in (b) in order for the second spectral sequence of the complex $F(L)$ to be the one in 2.4.2. We note once more that, if the degrees of K are positive, the two spectral sequences derived from F are *cohomological* spectral sequences. We could even define the spectral sequences derived from F on K if we no longer assume that K has degrees bounded below, *provided F has finite injective dimension*, i.e. $R^pF = 0$ for some p. This fact does not seem to be found in [6], but since we will not be using it subsequently, we confine ourselves to mentioning it here without proof. Of course we could also define the left derived spectral functors of F provided $\mathbf{C}$ has enough projectives, and we can also imagine the case of a contravariant functor. In [6], we find the case that F is a multifunctor but we will not require that case.

Let $\mathbf{C}, \mathbf{C}'$ and $\mathbf{C}''$ be three abelian categories. We consider two covariant functors $F : \mathbf{C} \longrightarrow \mathbf{C}'$ and $g : \mathbf{C}' \longrightarrow \mathbf{C}''$. We assume that every object of $\mathbf{C}$ and every object of $\mathbf{C}'$ is isomorphic to a subobject of an injective object, which allows us to consider the right derived functors of F, G, and $G\text{o}F$. We propose to establish the relation between these derived functors. Let $A \in \mathbf{C}$; let $C(A)$ be the complex associated with a right resolution of A by injectives. We consider the complex $F(C(A))$ in $\mathbf{C}'$. It is determined up to homotopy when we vary $C(A)$. It immediately follows that the spectral sequences defined by G and this complex $F(C(A))$ depend only on A. Thus these are cohomological spectral functors on C, converging to the same place, called *spectral functors of the composite functor $G\text{o}F$*. The formulas given above immediately give their initial terms:

$$\mathrm{I}_2^{p,q}(A) = (R^p((R^qG)F))(A) \qquad\qquad \mathrm{II}_2^{p,q}(A) = R^pG(R^qF(A))$$

The most important case by far for obtaining non-trivial spectral sequences is the one in which F transforms injectives into objects annihilated by the R^qG for $q \geq 1$ (such objects are called *G-acyclic*), and in which $R^0G = G$ (i.e. G is left exact); then $\mathrm{I}_2^{p,q} = 0$ if $q > 0$, and reduces to $R^p(GF)$ when $q = 0$ the result of which is that the common abutment of the two spectral sequences is identified with the right derived functor of GF. We thus obtain:

2.4.1 THEOREM. *Let $\mathbf{C}, \mathbf{C}'$ and $\mathbf{C}''$ be abelian categories. Assume that every object of $\mathbf{C}$ and every object of $\mathbf{C}'$ is isomorphic to a subobject of an injective. We consider the covariant functors $F : \mathbf{C} \longrightarrow \mathbf{C}'$ and $G : \mathbf{C}' \longrightarrow \mathbf{C}''$; we assume that G is left exact and that F transforms injectives into G-acyclic objects (i.e. annihilated by the R^qG for $q > 0$). Then there exists a cohomological spectral functor on $\mathbf{C}$, with values in $\mathbf{C}''$, converging to the right derived functor $\mathcal{R}(GF)$ of GF (appropriately filtered), and whose initial term is*

$$E_2^{p,q}(A) = R^pG(R^qF(A)) \tag{2.4.3}$$

Remarks 1. The second assumption about the pair (F, G) means that functors $(R^qG)F$ for $q > 0$ are effaceable (cf. 2.2), or even that for every $A \in \mathbf{C}$ we can find a monomorphism from A into an M such that $(R^qG)(F(M)) = 0$ for $q \geq 1$. This is how we will usually verify this hypothesis.

2. We immediately verify that in order to calculate the second spectral sequence of a composite functor (i.e. the one in question in Theorem 2.4.1), it is sufficient to take a resolution $C(A)$ of A by some C^i that are F-acyclic (and not necessarily injective) and to take the second hyperhomological spectral sequence of the functor G with respect to the complex $FC(A)$.

3. We note two important special cases in which one of the two hyperhomological spectral sequences of a functor F with respect to a complex K is degenerates *if K is a resolution of an object $A \in \mathbf{C}$, then $\mathcal{R}^nF(K) = R^nF(A)$* and thus the graded object $(R^nF(A))$ is the abutment of the spectral sequence with the initial term $\mathrm{I}_2^{p,q}(K) = H^p(R^qF(K))$. *If the K^n are F-acyclic (i.e. $R^mF(K^n) = 0$ for $m > 0$), then $\mathcal{R}^nF(K) = H^n(F(K))$*, so the graded object $(H^n(F(K)))$ is the abutment of a spectral sequence with the initial term $\mathrm{II}_2^{p,q}(K) = R^pF(H^q(K))$. Combining these two results, we find therefore: *If K is a resolution of A by F-acyclic objects, we have $R^nF(A) = H^n(F(K))$.* The isomorphisms thus obtained are moreover also the morphisms induced by a homomorphism from K into an injective resolution of A; they also coincide up to sign with the homomorphisms defined by the "iterated connecting homomorphism" in K [6, Chapter V.7].

2.5 Resolvent functors

Let $\mathbf{C}$ and $\mathbf{C}'$ be abelian categories. We suppose that every object of $\mathbf{C}$ is isomorphic to a subobject of an injective. Let F be a left exact covariant functor from $\mathbf{C}$ to $\mathbf{C}'$. We call a *resolvent functor* of F a covariant functor $\mathbf{F}$ defined on $\mathbf{C}$, with values in the category of complexes of positive degree in $\mathbf{C}'$. $\mathbf{F}(A) = (\mathbf{F}^n(A))$ equipped with an augmentation $F \longrightarrow \mathbf{F}$ (i.e. with a homomorphism from the functor $F \longrightarrow Z^0(\mathbf{F})$ of the 0-cocycles of $\mathbf{F}$), satisfying the following conditions: (i) the functor $\mathbf{F}$ is exact; (ii) $F \longrightarrow Z^0(\mathbf{F})$ is an isomorphism; (iii) if A is injective, $\mathbf{F}(A)$ is acyclic in degrees > 0.

Let $\mathbf{F}$ be a resolvent functor for F. We consider the functors $H^n\mathbf{F}(A)$ on $\mathbf{C}$, with values in $\mathbf{C}'$. Because of condition (i), they form a cohomological functor; because of (ii) it reduces to F in dimension 0; because of (iii) the $H^n\mathbf{F}(A)$ for $n > 0$ are effaceable. Consequently:

2.5.1 Proposition. *If $\mathbf{F}$ is a resolvent functor for F, then for every $A \in \mathbf{C}$ we have unique isomorphisms $H^n\mathbf{F}(A) = R^nF(A)$, defining an isomorphism of cohomological functors, and reducing in dimension 0 to the augmentation isomorphism.*

We are going to give another proof of the preceding proposition allowing an easy computation of these isomorphisms:

2.5.2 Proposition. *Let $\mathbf{F}$ be a resolvent functor for F. Let $A \in \mathbf{C}$ and let $C = (C^p(A))$ be a right resolution of A by F-acyclic objects. We consider the bicomplex $\mathbf{F}C(A) = (\mathbf{F}^qC^p(A))_{p,q}$ and the natural homomorphisms*

$$F(C(A)) \longrightarrow \mathbf{F}(C(A)) \longleftarrow \mathbf{F}(A) \tag{2.5.1}$$

defined respectively by the augmentation $F \longrightarrow \mathbf{F}$ *and by the augmentation* $A \longrightarrow C(A)$. *Then the corresponding homomorphisms*

$$H^n F(C(A)) = R^n F(A) \longrightarrow H^n \mathbf{F}(C(A)) \longleftarrow H^n \mathbf{F}(A) \tag{2.5.2}$$

are isomorphisms, and the corresponding isomorphism $H^n \mathbf{F}(A) \cong R^n F(A)$ *is the one in Proposition 2.5.1.*

We consider $F(C(A))$ (respectively $\mathbf{F}(A)$) as a bicomplex whose second (respectively first) degree is 0. The first spectral sequence of $\mathbf{F}(C(A))$ has as its initial term $H(F(C(A)))$ (since $H^q_{\mathrm{II}}(\mathbf{F}(C(A)))$ vanishes for $q > 0$ and can be identified as $F(C(A))$ for $q = 0$, from Proposition 2.5.1 and the fact that the $C^p(A)$ are F-acyclic), hence the first homomorphism (2.5.1) induces an isomorphism for the initial terms of the spectral sequence I and therefore induces an isomorphism on cohomology. Similarly, $H_{\mathrm{I}}(\mathbf{F}(C(A)))^{p,q}$ vanishes for $p > 0$ and equals $\mathbf{F}(A)$ for $p = 0$ (since $\mathbf{F}$ is an exact functor and $C(A)$ is a resolution of A), therefore the second homomorphism (2.5.1) induces an isomorphism of the initial terms of the spectral sequences II, and therefore also induces an isomorphism of the cohomology. To show that the isomorphism obtained from $H^n \mathbf{F}(A)$ to $R^n F(A)$ is really the one in Proposition 2.5.1, we can take an *injective* resolution $C'(A)$ of A and a homomorphism from $C(A)$ to $C'(A)$, and imagine the associated homomorphism from diagram (2.5.2) to the analogous diagram for $C'(A)$, which shows us that the isomorphism obtained is independent of the choice of $C(A)$. If we limit ourselves from now on to injective resolutions, we see immediately that the isomorphisms obtained are functorial and commute with coboundary homomorphisms; moreover, they reduce in dimension 0, to the augmentation isomorphism (or rather, its inverse), and therefore these isomorphisms are really those of Proposition 1.

Examples. (a) We consider the identity functor $I : \mathbf{C} \longrightarrow \mathbf{C}$. A resolvent functor of I (also called *identity resolution* in $\mathbf{C}$) is therefore an *exact* covariant functor C from $\mathbf{C}$ to the category of complexes of positive degrees in $\mathbf{C}$, equipped with an augmentation $A \longrightarrow C(A)$ that is an isomorphism from A to $Z^0C(A)$, and such that $C(A)$ is a resolution of A if A is injective; $C(A)$ is a resolution of A, no matter what A is, according to Proposition 2.5.1 (since the functor I is exact, we have $R^n I = 0$ for $n > 0$). Let C be a resolution of the identity in $\mathbf{C}$ and let F be a left exact covariant functor from $\mathbf{C}$ to $\mathbf{C}'$; *we assume* $R^n F(C^i(A)) = 0$ for all $n > 0$ (i.e. the $C^i(A)$ are F-acyclic). *Then* $\mathbf{F}(A) = F(C(A))$ *is a resolvent functor for* F. In effect, this functor is exact, since if we have an exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, we conclude from this the exactness of $0 \longrightarrow C^i(A') \longrightarrow C^i(A) \longrightarrow C^i(A'') \longrightarrow 0$, from which, since $R^i F(C^i(A')) = 0$, we get an exact sequence $0 \longrightarrow F(C^i(A')) \longrightarrow F(C^i(A)) \longrightarrow F(C^i(A'')) \longrightarrow 0$. Thus (i) is proved; similarly (ii) is proved since F is left exact and C is exact; and (iii) is proved since we have ($C(A)$ being an F-acyclic resolution of A) $H^n(C(A)) = R^n F(A)$, which vanishes if $n > 0$ and A is injective. A convenient way to construct a resolution of the identity in $\mathbf{C}$ is to take an exact functor $C^0(A) : \mathbf{C} \longrightarrow \mathbf{C}$ and a functorial *monomorphism* $A \longrightarrow C^0(A)$; we then define by recursion the $C^n(A)$ $(n \geq 0)$ and the homomorphisms $d^{n-1} : C^{n-1} \longrightarrow C^n$ taking $C^1(A) = C^0(C^0(A)/\operatorname{Im}(A))$, and for $n \geq 2$, $C^n(A) = C^0(C^{n-1}(A)/\operatorname{Im} C^{n-2}(A))$, d^{n-1} being defined with the help of the augmentation homomorphism of $Q = C^{n-1}(A)/\operatorname{Im} C^{n-2}(A)$ in $C^0(Q)$. We thus obtain a resolution of the identity; for the $C^i(A)$ to be F-acyclic, it is sufficient for $C^0(A)$ to be.

(b) Let $P = (P_n)$ be a projective resolution of an object A of $\mathbf{C}$. Let F be the functor $F(B) = \operatorname{Hom}(A, B)$ from $\mathbf{C}$ to the category of abelian groups. Then F admits the resolvent functor $\operatorname{Hom}(P, B)$.

Now we will show that the most important spectral sequences can be calculated using resolvent functors. Let $\mathbf{F}$ be a resolvent functor of a left exact functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$. Let K be a complex in $\mathbf{C}$ with degrees bounded on the left. Consider the bicomplex $\mathbf{F}(K) = (\mathbf{F}^q(K^p))_{p,q}$, which depends functorially on K. So do its spectral sequences and their abutments. The initial terms of these two spectral sequences can easily be written using the exactness of the functor $\mathbf{F}$ and Proposition 2.5.1; we find

$$\mathrm{I}_2^{p,q}(\mathbf{F}(K)) = H^p(R^qF(K)) \qquad \mathrm{II}_2^{p,q}(\mathbf{F}(K)) = R^pF(H^q(K)) \tag{2.5.3}$$

These are natural transformations with the initial terms of the hyperhomology spectral sequences (2.4.2) of F with respect to the complex K.

2.5.3 Proposition. *The isomorphisms (2.5.3) come from natural equivalences of the first, respectively the second, spectral sequence of the bicomplex* $\mathbf{F}(K)$ *with the first, respectively second, spectral sequence of hyperhomology of the functor* F *with respect to the complex* K.

These isomorphisms will be made explicit in the following proof. Let $C(K) = (C^{p,q}(K))_{p,q}$ be an "injective resolution" of K in the sense of [6, Chapter XVII]; considering K as bicomplex whose second degree is zero, we have an augmentation homomorphism $K \longrightarrow C(K)$. Consider the bicomplex $M(K) = \mathbf{F}(C(K))$ given by

$$M^{p,q}(K) = \sum_{q'+q''=q} \mathbf{F}^{q''}(C^{p,q'}(K))$$

$M(K)$ is not uniquely determined by K; However, the two spectral sequences of $M(K)$ and their common abutments are uniquely determined (since $C(K)$ is determined up to homotopy equivalence of bicomplexes); these are well-determined spectral functors in the variable K. Set $L(K) = F(C(K))$; the augmentation homomorphisms $K \longrightarrow C(K)$ and $F \longrightarrow \mathbf{F}$ define homomorphisms of bicomplexes

$$L(K) = F(C(K)) \longrightarrow M(K) = \mathbf{F}(C(K)) \longleftarrow \mathbf{F}(K)$$

whence *natural transformations* for the corresponding spectral sequences (independent of the choice of $C(K)$):

$$\begin{array}{c} \mathrm{I}\,L(K) \longrightarrow \mathrm{I}\,M(K) \longleftarrow \mathrm{I}\,\mathbf{F}(K) \\ \mathrm{II}\,L(K) \longrightarrow \mathrm{II}\,M(K) \longleftarrow \mathrm{II}\,\mathbf{F}(K) \end{array} \tag{2.5.4}$$

defining the same homomorphism for the abutments

$$HL(K) \longrightarrow HM(K) \longleftarrow H\mathbf{F}(K)$$

For the initial terms, the homomorphisms (2.5.4) give:

$$\begin{array}{c} \mathrm{I}_2^{p,q}\, L(K) \longrightarrow \mathrm{I}_2^{p,q}\, M(K) \longleftarrow \mathrm{I}_2^{p,q}\, \mathbf{F}(K) \\ \mathrm{II}_2^{p,q}\, L(K) \longrightarrow \mathrm{II}_2^{p,q}\, M(K) \longleftarrow \mathrm{II}_2^{p,q}\, \mathbf{F}(K) \end{array} \tag{2.5.5}$$

We will prove that the homomorphisms are isomorphisms, and that the corresponding isomorphisms between the extreme terms of (2.5.5) (which are the initial terms of the hyperhomology spectral sequences of F for K, respectively for the bicomplex $\mathbf{F}(K)$) are those resulting from the formulas (2.5.3). It follows that the homomorphisms (2.5.4) are isomorphisms, whence isomorphisms between the extreme terms; these will obviously be the desired isomorphisms. Everything comes down to proving that the middle terms of lines (2.5.5) have, respectively, the form $H^p(R^q(F(K)))$ and $R^pF(H(K))$ (the proof that the homomorphisms in (2.5.5) are in, fact, natural equivalences is purely mechanical, and moreover, implicitly contained in the following argument).

We will show that $H_{\mathrm{II}}(M^{p,q}) = R^qF(K^p)$ (from which we immediately see that $\mathrm{I}_2^{p,q}(M) = H^p(R^qF(K)))$. Thus for fixed p, $M^(p*) = (M^{p,q})_q$ is the simple complex associated to the bicomplex $(\mathbf{F}^{i''}(C^{p,q'}(K))_{q',q''}) = \mathbf{F}(C(K^p))$, where $C(K^p)$ denotes the complex $C^{p*}(K)$, which is an injective resolution of K^p. It follows that $H^q(M^{p*}) = H^q(\mathbf{F}(C(K^p)))$, and the last term is identified with $R^qF(K^p)$ by virtue of Proposition 2.5.2.

It remains to calculate the middle term of the second line (2.5.5). First we have

$$H_{\mathrm{I}}(M)^{p,q} = \sum_{q'+q''=q} H^p(\mathbf{F}^{q''}(C^{*q'}(K)))$$

Since $\mathbf{F}^{q''}$ is an exact functor we get for the general term of the sum of the second term $\mathbf{F}^{q''}(H^p(C^{*q'}(K))) = \mathbf{F}^{q''}(C^{q'}(H^p(K)))$, where $C(H^p(K))$ denotes an injective resolution of $H^p(K)$ (recall the definition of an "injective resolution" $C(K)$ of the complex K!). We therefore find $(H_{\mathrm{II}}(H_{\mathrm{I}}(M)))^{p,q} = H^q(\mathbf{F}(C(H^p(K))))$, which is identified with $R^qF(H^p(K))$ by virtue of Proposition 2.5.2, Q.E.D.

Since $\mathbf{F}$ is always a resolvent functor for the left exact functor $F\mathbf{C} \longrightarrow \mathbf{C}'$, we assume in addition that we have a covariant functor G from $\mathbf{C}'$ to an abelian category $\mathbf{C}''$, and that in $\mathbf{C}'$ every object is isomorphic to a subobject of an injective. For every $A \in \mathbf{C}$ we consider the second hyperhomology spectral sequence of the functor G with respect to the complex $\mathbf{F}(A)$; it is a spectral functor in A whose initial term, by virtue of Proposition 2.5.1, can be readily calculated:

$$\mathrm{II}_2^{p,q}\, G(\mathbf{F}(A)) = R^pG(R^qF(A)) \tag{2.5.6}$$

This is a natural transformation with the initial term of the second spectral sequence of the composite functor GF.

2.5.4 Proposition. *The isomorphism (2.5.6) comes from a natural transformation of the second hyperhomology spectral sequence of G with respect to the complex $\mathbf{F}(K)$, to the second spectral sequence of the composite functor GF.*

Let $C(A)$ be an injective resolution of A. Consider $\mathbf{F}(C(A))$ to be a single complex in $\mathbf{C}'$, and consider the natural homomorphisms, 2.5.1, whence we get corresponding homomorphisms for the second spectral sequences of hyperhomology of G:

$$\mathrm{II}\, G(F(C(A))) \longrightarrow \mathrm{II}\, G(\mathbf{F}(C(A))) \longleftarrow \mathrm{II}\, G(\mathbf{F}(A))$$

Again everything comes back to showing that these are isomorphisms, and to do so, showing that the corresponding homomorphisms for the initial terms are isomorphisms. Now by virtue of Proposition 2.5.1, the homomorphisms $H(F(C(A))) \longrightarrow H(\mathbf{F}(C(A))) \longleftarrow H(\mathbf{F}(A))$ are isomorphisms, from which the desired conclusion is readily obtained.

Let $\mathbf{C}, \mathbf{C}', \mathbf{C}''$ and $\mathbf{C}'''$ be abelian categories. For the first three, we assume that every object of each is isomorphic to a subobject of an injective. Consider the covariant functors F, G, F', G' (see the diagram below).

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "G'"] \arrow[d, "{F,\mathbf{F}}"'] & \mathbf{C}''' \arrow[d, "{F',\mathbf{F}'}"] \\
\mathbf{C}' \arrow[r, "G"'] & \mathbf{C}''
\end{tikzcd}$$

We take as given a resolvent functor $\mathbf{F}$ for F and a resolvent functor $\mathbf{F}'$ for F' satisfying the commutativity conditions $\mathbf{F}'G' = G\mathbf{F}$.[7] In addition we assume that $\mathbf{F}$ sends injective objects to G-acyclic objects.

2.5.5 Proposition. *Assume the preceding conditions. For any $A \in \mathbf{C}$, consider the two hyperhomology spectral sequences of G with respect to the complex $\mathbf{F}(A)$; these are the spectral functors in A. They are isomorphic respectively to the second spectral functor of the composite functors $F'G'$ and GF.*

The assertion about the composite functor GF is nothing but Proposition 2.5.4. As for the second spectral sequence of the composite functor $F'G'$, it is by definition the second spectral sequence of F' with respect to the complex $G'(C(A))$ (where $C(A)$ denotes an injective resolution of A), therefore is identified, by virtue of Proposition 2.5.3 with the second spectral sequence of the bicomplex $\mathbf{F}'G'(C(A))$ (whose first degree is the one that comes from $C(A)$); thus we have $\mathbf{F}'G' = G\mathbf{F}$. It is therefore necessary to calculate the

[7]It is understood that this isomorphism is natural and respects the coboundary homomorphism in the complexes $\mathbf{F}'G'(A)$ and $G\mathbf{F}(A)$.

spectral sequence II($G\mathbf{F}(C(A))$). Thus for any fixed q, $\mathbf{F}^q C(A)$ is a resolution of $\mathbf{F}^q(A)$ (since $\mathbf{F}^q$ is exact) by G-acyclic objects, therefore $\mathbf{F}(C(A))$ becomes, by exchanging its two degrees, a resolution of $\mathbf{F}(A)$ by G-acyclic objects. It follows from 2.4 that the second spectral sequence of $G\mathbf{F}(C(A))$ can be identified with the first hyperhomology spectral sequence of G with respect to the complex $\mathbf{F}(A)$.

Corollary. *If $\mathbf{C}, \mathbf{C}', \mathbf{C}'', \mathbf{C}'''$ and F, G, F', G' are as above (see diagram), we assume as given resolvent functors $\mathbf{F}, \mathbf{G}, \mathbf{F}', \mathbf{G}'$ for F, G, F', G'. We assume $\mathbf{F}'^i \mathbf{G}'^j = \mathbf{G}^j \mathbf{F}^i$ (natural transformations compatible with the boundary operators in the resolving functor), and we also assume that $\mathbf{F}$ (respectively, $\mathbf{G}'$) transforms injective objects into objects that are G-acyclic (respectively F'-acyclic). Then for every $A \in \mathbf{C}$, the second spectral sequences of the composite functors $F'G'$ and GF for A are identified with the first and second hyperhomologyspectral sequences of G for the complex $\mathbf{F}(A)$, or with the first and second hyperhomology spectral sequence of F' for the complex $\mathbf{G}'(A)$, or with one or the other of the two spectral sequences of the bicomplex $\mathbf{G}(\mathbf{F}(A)) = \mathbf{F}'(\mathbf{G}'(A))$.*

Chapter 3

Cohomology with coefficients in a sheaf

3.1 General remarks on sheaves

Let X be a topological space (not necessarily separated). Recall (1.7, Example H) that we call *presheaves of sets* on X any inductive system of sets defined on the open non-empty subsets of X, ordered by $\supseteq$. A presheaf consists, therefore, in giving for every open $U \subseteq X$, a set $F(U)$, and for every open non-empty pair U, V with $U \supseteq V$ of a *restriction function* $\phi_{V,U} : F(U) \longrightarrow F(V)$, such that $\phi_{U,U}$ is the identity function of $F(U)$ and $\phi_{W,V}\phi_{V,U} = \phi_{W,U}$ if $U \supseteq V$
contW. We say that the presheaf F is a *sheaf* provided for every cover (U_i) of an open set $U \subseteq X$ by non-empty sets, and every family (f_i) of elements $f_i \in FU_i$ such that $\phi_{U_{ij}U_i} f_i = \phi_{U_{ij}U_j} f_j$ for each pair (ij) such that $U_{ij} = U_i \cap U_j \neq \emptyset$, there exists a unique $f \in F(U)$ such that $\phi_{U_iU} f = f_i$ for every i. If in the preceding definition, we assume that the $F(U)$ are groups (respectively rings, etc.) and that the ϕ_{VU} are homomorphisms, we obtain the notion of *presheaf* or *sheaf* of groups, (respectively of *rings*, etc.); more generally we might define the notion of presheaf or sheaf with values in a given category (cf. 1.1). The presheaves or sheaves on X with values in a given category form a category, the morphisms being defined as morphisms of inductive systems. The presheaves or sheaves on X with values in an *additive* category, for example the category of abelian groups, form an additive category, and in the case of presheaves and sheaves of abelian groups, even an abelian category. (For the sake of brevity, we will say *abelian sheaf* and *abelian presheaf* for a sheaf or presheaf of abelian groups). But we will take care that the identity functor, which associates to an abelian sheaf the corresponding abelian presheaf, is left exact but not exact: if we have a homomorphism of sheaves $u : F \longrightarrow G$ its cokernel as a homomorphism of presheaves is the presheaf $Q(U) = G(U)/\operatorname{Im} F(U)$, which in general is not a sheaf; its

cokernel as a homomorphism of sheaves is the sheaf *associated* with the presheaf Q (see below). We will cease emphasizing these questions, which are already rather well known (cf. [4] and Godement's book [9]).

Let F be a presheaf of sets on X. We set for every $x \in X$: $F(x) = \varinjlim F(U)$, the inductive limit being taken over the filtered set of open neighborhoods U of x. On the set $\overline{F} = \bigcup F(x)$, we put the topology generated by the set of subsets of F that have the form $A(f)$ where, for every open set $U \subseteq X$ and every $f \in F(U)$ we denote by $A(f)$ the set of canonical images $f(x)$ of f in $F(x)$, for $x \in U$. When $\overline{F}$ is equipped with this topology, the natural function of $\overline{F}$ to X is a local homeomorphism (i.e. every point in $\overline{F}$ has an open neighborhood mapped homeomorphically on an open set of X) and we say (following Godement) that $\overline{F}$ is the total space over X. Moreover, a total space over X, called E, defines a sheaf $\mathcal{F}(E)$ in a natural way, namely the one that to an open set U associates the set of continuous sections of E over U; moreover, the total space associated to $\mathcal{F}(E)$ can be canonically identified with E. If we begin with a presheaf F on X we have, moreover, a canonical homomorphism $F \longrightarrow \mathcal{F}(F)$, since every $f \in F(U)$ defines a continuous section, $x \mapsto f(x)$ of F over U; this homomorphism is an isomorphism if and only if F is a sheaf. These considerations prove: (1) *the notion of sheaf of sets on X is equivalent to the notion of total space over X* (specifically, we have defined an equivalence of the category of sheaves of sets on X with the category of total spaces over X). (2) *To every presheaf F on X there corresponds a sheaf $\mathcal{F}(\overline{F})$ and a homomorphism $F \longrightarrow \mathcal{F}(\overline{F})$, which is an isomorphism if and only if F is a sheaf* (moreover, $\mathcal{F}(\overline{F})$ is a *functor* in F, and the homomorphism $F \longrightarrow \mathcal{F}(\overline{F})$ is natural in F). If we wish to interpret the notion of sheaf of groups (or of sheaf of abelian groups, etc.) in terms of total spaces, we have to give on each stalk of the total space a group law (respectively, abelian group law, etc.) so as to satisfy a natural continuity condition: we then recover the definition of [4, XIV]. We see immediately on the corresponding total spaces when a homomorphism $F \longrightarrow G$ of sheaves is a monomorphism or an epimorphism: it is necessary and sufficient that on each stalk $F(x)$ the corresponding homomorphism $F(x) \longrightarrow G(x)$ is a monomorphism or an epimorphism, respectively. Similarly, the stalk over x of the kernel, cokernel, image, coimage of a homomorphism of sheaves of abelian groups is obtained by taking the kernel, cokernel, etc. of the homomorphism of abelian groups $F(x) \longrightarrow G(x)$.

Let $\mathbf{O}$ be a sheaf of unital rings on X. A *sheaf of left* $\mathbf{O}$*-modules* or in short, a *left* $\mathbf{O}$*-module* on X is a sheaf of abelian groups F together with the giving for each open $U \subseteq X$ of a structure of a left unitary $\mathbf{O}(U)$-module structure on $F(U)$ such that the restriction operations $F(U) \longrightarrow F(V)$ are compatible with the operations of $\mathbf{O}(U)$ and $\mathbf{O}(V)$ on $F(U)$, respectively $F(V)$ (we can also express this by saying that we are given a homomorphism of the sheaf of rings $\mathbf{O}$ into the sheaf of rings of the germs of endomorphisms of the sheaf F of abelian groups). We define right $\mathbf{O}$-modules similarly; for the sake of brevity we will simply say $\mathbf{O}$-module instead of left $\mathbf{O}$-module. The notion of homomorphism of $\mathbf{O}$-modules and of the composition and addition of such homomorphisms is obvious. We then obtain *the additive category of* $\mathbf{O}$*-modules on* X*, denoted* $\mathbf{C}^{\mathbf{O}}$. If, for example, k is a unital ring, we can consider on X the corresponding constant sheaf of rings, denoted k_X. The category of k_X-modules is none other than the category of sheaves of k-modules on X; if, for example, $k = \mathbf{Z}$, we get the category of sheaves of abelian groups. We recall the following fact, mentioned in passing in 1.5 and 1.9:

3.1.1 Proposition. *Let* $\mathbf{O}$ *be a sheaf of unital rings on a space* X*. Then the additive category* $\mathbf{C}^{\mathbf{O}}$ *of* $\mathbf{O}$*-modules on* X *is an abelian category satisfying Axioms* AB 5) *and* AB 3*)*, and admits a admits a generator.*

We should note that the direct sum S of a family (F_i) of sheaves F_i is constructed simply by taking for each open U the direct sum of the $F_i(U)$ and passing to the sheaf associated to the just-constructed presheaf; the process is the same for the construction of the product P of sheaves F_i. The essential difference between the two cases is that for every $x \in X$, $S(x)$ is indeed the direct sum of the $F_i(x)$, but $P(x)$ is not the direct product of the $F_i(x)$. We can readily see that Axiom AB 4$*$) is not satisfied in general (for example, by taking for $\mathbf{O}$ the constant sheaf $\mathbf{Z}_X$). Finally, recall that if, for any open $U \subseteq X$ we denote by $\mathbf{O}_U$ the sheaf of $\mathbf{O}$-modules whose restriction to $\complement U$ is zero and which coincides with $\mathbf{O}$ on U [4, XVII, Proposition 1], the family of $\mathbf{O}_U$ is a family of generators of $\mathbf{C}^{\mathbf{O}}$ which follows easily from Definition 1.9 and from the fact that the homomorphisms of $\mathbf{O}_U$ to an $\mathbf{O}$-module F are identified with the elements of $F(U)$. Taking into account Theorem 1.10.1, we find:

Corollary. *Every* $\mathbf{O}$*-module is isomorphic to a sub-*$\mathbf{O}$*-module of an injective* $\mathbf{O}$*-module.*

We indicate a direct proof, due to Godement, of this corollary. For every $x \in X$, let M_x be an $\mathbf{O}_x$-module and let M be the sheaf on X defined by $M(U) = \prod_{x \in X} M_x$; the restriction functions and the operations of $\mathbf{O}(U)$ on $M(U)$ are defined in the obvious way. M is an $\mathbf{O}$-module on X, by construction isomorphic to the product of the $\mathbf{O}$-modules M^x $(x \in X)$ obtained by defining $M^x(U) = M_x$ for $x \in U$ and zero otherwise. From this remark, we immediately deduce that for every $\mathbf{O}$-module F, the homomorphisms from F to M are identified with the families $(u_x)_{x \in X}$, where for every $x \in X$, u_x is an $\mathbf{O}(x)$-homomorphism from $F(x)$ to M_x. From this we conclude:

3.1.2 Proposition. *If for every* $x \in X$*,* M_x *is an injective* $\mathbf{O}_x$*-module then the product sheaf* M *defined above is an injective* $\mathbf{O}$*-module.*

Let F be an arbitrary $\mathbf{O}$-module. It is classic (and, moreover, a consequence of Theorem 1.10.1) that for every $x \in X$, $F(x)$ can be embedded an injective $\mathbf{O}_x$-module, namely M_x, it follows that we get an embedding of F into the injective $\mathbf{O}$-module M defined by the M_x.

We also point out for later use:

3.1.3 PROPOSITION. *Let M be an injective $\mathbf{O}$-module on X, U be an open subset of X, and $\mathbf{O}_U$, (respectively M_U be the restriction of $\mathbf{O}$, (respectively M) to U. Then M_U is an injective $\mathbf{O}_U$-module.*

M_U is clearly an $\mathbf{O}_U$-module. Let F be an $\mathbf{O}_U$-module, G a submodule, and u a homomorphism of G to M_U. We prove that u can be extended to a homomorphism from F to M_U. For every $\mathbf{O}_U$-module H on U, let $\overline{H}$ be the $\mathbf{O}$-module obtained, in terms of total spaces, by "extending H by zero on $\complement U$", cf. [4, Chapter XVII, Proposition 1]. Then giving a homomorphism of $\mathbf{O}_U$-modules $u : G \longrightarrow M_u$ is equivalent to giving a homomorphism of $\mathbf{O}$-modules $\overline{G} \longrightarrow M$ since $\overline{G}$ is a submodule of $\overline{F}$ and M is injective, $\overline{u}$ can be extended to a homomorphism of $\overline{G}$ to M, which induces the desired homomorphism of G to M_U. We note that Proposition 3.1.3 becomes *false* if we assume that U is *closed* rather than open. We prove in a completely analogous way:

3.1.4 PROPOSITION. *Let M be an injective $\mathbf{O}$-module on a closed subset Y of X. Then the $\mathbf{O}$-module M^X that coincides with M on Y and is zero on its complement, is injective.*

3.2 Definition of the $H^p_\Phi(X,F)$

Let X be a topological space. We denote by $\mathbf{C}^X$ the abelian category $\mathbf{C}^{\mathbf{Z}}_X$ of abelian sheaves on X. If F is such a sheaf and A is a subset of X, we denote by $\Gamma(A,F)$ the group of sections of F (considered to be a total space) over A, and we set $\Gamma(F) = \Gamma(X,F)$ so that we have $\Gamma(A,F) = \Gamma(F|A)$ where $F|A$ indicates the restriction of F to A. More generally, let Φ be an increasing filter of closed non-empty subsets of X, such that $A \in \Phi$, $B \subseteq A$ implies that $B \in \Phi$. For the sake of brevity, we say that Φ is an *cofilter of closed subsets* of X. We denote by $\Gamma_\Phi(F)$ the subgroup of $\Gamma(F)$ consisting of sections f whose support (the complement of the largest open set in X on which the restriction of f vanishes) is an element of Φ. We see immediately that $\Gamma_\Phi(F)$ is a left exact functor defined on $\mathbf{C}^X$ with values in the category of abelian groups. We must consider its right derived functors to be identical to the right satellites (cf. 2.3) which exists by virtue of the corollary to Proposition 3.1.1. We denote them by $H^p_\Phi(X,F)$ (where p is an arbitrary integer). According to the theory:

3.2.1 PROPOSITION. *The system of functors $H^p_\Phi(X,F)$ ($-\infty < p < +\infty$) is characterized by the following conditions: they form a cohomological functor on $\mathbf{C}^X$ with values in the category of abelian groups, and $H^p_\Phi(X,F)$ vanishes for $p < 0$, coincides with $\Gamma_\Phi(F)$ for $p = 0$ and is effaceable, that is, vanishes when F is injective, for $p > 0$.*

To calculate the $H^p_\Phi(X,F)$ we take a right resolution of F by injective sheaves C^i or, more generally, such that we know in advance that the $H^p_\Phi(X,C^i) = 0$ for $p > 0$ (we then the say that the C^i are Γ_Φ-acyclic), we consider the complex $C(F)$ formed by the C^i, and we get

$$H^p_\Phi(X,F) = H^p\Gamma_\Phi(C(F))$$

To apply this method, it is thus important to know the criteria that allow us to assert that a sheaf is Γ_Φ-acyclic; we will indicate them below (cf. 3.3).

Let f be a continuous function from a space X to a space Y. For every sheaf F on Y, we define in a natural way the *inverse image sheaf* of F under f (denoted $f^{-1}(F)$ by abuse of notation): if we consider F as a total space over Y, it is sufficient to use the definition of the inverse image of a fibered space.[p] We thus obtain an *exact additive covariant functor* $F \longrightarrow f^{-1}(F)$ from $\mathbf{C}^Y$ to $\mathbf{C}^X$, which we are going to call g. For every $F \in \mathbf{C}^Y$, we have an obvious homomorphism $\Gamma(F) \longrightarrow \Gamma(g(F))$, which moreover is functorial, that is, we have a natural transformation $\Gamma^Y \longrightarrow \Gamma^X g$ (where we have put as the exponent the space relative to which we consider the functor Γ). More generally, consider a cofilter Φ of closed subsets of X and a cofilter Ψ of closed subsets of Y such that for every $B \in \Psi$, we have $f^{-1}(B) \in \Phi$. Then for every $f \in \mathbf{C}^Y$, the homomorphism $\Gamma(F) \longrightarrow \Gamma(g(F))$ applies $\Gamma_\Psi(F)$ to $\Gamma_\Phi(g(F))$, whence a natural transformation $\Gamma_\Psi \longrightarrow \Gamma_\Phi g$. Moreover, since g is an *exact* functor, the functor $(R\Gamma_\Phi)g$ can be considered to be a *cohomological functor* on $\mathbf{C}^Y$, reducing to $\Gamma_\Phi^X g$ in dimension 0; since the cohomological functor $R\Gamma_\Psi^Y$ is "universal" (Proposition 2.2.1), the natural transformation $\Gamma_\Psi^Y \longrightarrow \Gamma_\Phi^X g$ extends uniquely to a natural transformation of cohomological functors $R\Gamma_\Psi \longrightarrow (R\Gamma_\Phi)g$. Thus we have proved:

3.2.2 Proposition. *Let f be a continuous function from a space X to a space Y and Φ (respectively Ψ) be a cofilter of subsets of X (respectively Y), such that $B \in \Psi$ implies $f^{-1}(B) \in \Phi$. We can find, for any sheaf F of abelian groups on Y, unique homomorphisms*

$$H^p_\Psi(Y, F) \longrightarrow H^p_\Phi(X, f^{-1}(F)), \qquad -\infty < p < +\infty$$

in order to get a natural transformation of cohomological functors that reduces to the natural homomorphism in dimension 0.

These homomorphisms will be called the natural homomorphisms. From their uniqueness there results an obvious property of transitivity, whose formulation is left to the reader.

In particular, if Y is a *subset* of X, and if we set $H^p_\Psi(Y, F) = H^p_\Psi(Y, F|Y)$, where $F|Y$ indicates the "restriction" of the sheaf F to Y, we have "*restriction* homomorphism" $H^p_\Phi(X, F) \longrightarrow H^p_{\Phi\cap Y}(Y, F)$, where $\Phi \cap Y$ indicates the *trace* of Φ on Y.

3.3 Criteria for Acyclicity

What we develop in this section, which will subsequently be very useful, is attributable to Godement and will be dealt with in detail in the book by Godement mentioned in the introduction, [9].

[p]Translator's note: Now known as a fibered product or pullback.

3.3.1 LEMMA. *Let F be a covariant functor from an abelian category $\mathbf{C}$ to another one $\mathbf{C}'$. We assume that every object of $\mathbf{C}$ is isomorphic to a subobject of an injective object. Let $\mathbf{M}$ be a class of objects of $\mathbf{C}$ that satisfies the following conditions: (i) for every element $A \in \mathbf{C}$, there is a monomorphism from A to an $M \in \mathbf{M}$; (ii) every $A \in \mathbf{C}$ that is isomorphic to a direct factor of an $M \in \mathbf{M}$ belongs to $\mathbf{M}$; (iii) for every exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathbf{C}$, if M' and M belong to $\mathbf{M}$, then $M'' \in \mathbf{M}$, and the sequence $0 \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0$ is also exact. Under these conditions, every injective object of $\mathbf{C}$ belongs to $\mathbf{M}$, and for every $M \in \mathbf{M}$, we have $R^pF(M) = 0$ for $p > 0$.*

First let I be an injective object in $\mathbf{C}$. I can be embedded into an $M \in \mathbf{M}$ by virtue of (i), and is thus isomorphic to a direct factor of M (since I is injective) and hence belongs to $\mathbf{M}$, by virtue of (ii). Let $M \in \mathbf{M}$. We show that $R^pF(M) = 0$ for $p > 0$. To see this, we consider a right resolution of M by injective objects C^i ($i \geq 0$). Let Z^i be the subobject of cycles in C^i. It suffices to prove that the sequences $0 \longrightarrow F(M) \longrightarrow F(C^0) \longrightarrow F(Z^1) \longrightarrow 0$, $0 \longrightarrow F(Z^0) \longrightarrow F(C^0) \longrightarrow F(Z^1) \longrightarrow 0$, etc. are all exact, and for this it suffices to prove by virtue of (iii) that the Z^i and C^i are in $\mathbf{M}$. We know this already for the C^i because they are injective, and for the Z^i, it follows from (iii) by induction on i.

COROLLARY. *For every $A \in \mathbf{C}$, we can calculate the $R^pF(A)$ with the aid of an (arbitrary) resolution of A by $C^i \in \mathbf{M}$.*

In effect, such a resolution exists by virtue of (i), making it possible to calculate the $R^pF(A)$ since the C^i are F-acyclic by virtue of the lemma.

A sheaf F of sets on a space X is said to be *flabby* (respectively *soft*) if for every open (respectively closed) subset $A \subseteq X$ every section of F over A is the restriction of a section of F over X. If for every $x \in X$ we are given a set E_x, the sheaf E whose set of sections over an open set $U \subseteq X$ is given by $E(U) = \prod_{x \in U} E_x$ (with the obvious restriction functions; compare the example dealt with before Proposition 3.1.2) is obviously both flabby and soft. We thereby conclude that every sheaf of sets can be embedded in a flabby sheaf of sets; similarly, every sheaf of abelian groups (respectively of $\mathbf{O}$-modules, if $\mathbf{O}$ is a sheaf of rings on X) can be embedded in a flabby sheaf of abelian groups (respectively, $\mathbf{O}$-modules). We should note that if a closed subset $A \subseteq X$ admits a paracompact neighborhood, then any section on A of a sheaf F defined on X is the restriction of a section defined in a neighborhood of A. It follows immediately that *if X is paracompact, a flabby sheaf is soft.* Now let Φ be a family of closed subsets of X satisfying the general conditions set forth in 3.2. A sheaf F of abelian groups on X is called Φ-*soft* if for every $A \in \Phi$ and every section of F over A, there is an $f \in \Gamma_\Phi(F)$ inducing the given section. We say that Φ is a *paracompatifying family* if, in addition to the conditions already set forth, it satisfies the following additional conditions (first introduced in [4]): *every $A \in \Phi$ is paracompact and has a neighborhood $B \in \Phi$.* We can easily see, as above, *that a flabby sheaf of abelian groups* *is Φ-soft for every paracompactifying family* Φ, whence it follows in particular that every sheaf of abelian groups can be embedded in a Φ-soft sheaf (since it can even be embedded in a flabby sheaf). The preceding definitions are of interest because of following:

3.3.2 PROPOSITION. *Let X be a topological space equipped with a "family Φ". We consider the functor Γ_Φ defined on the category $\mathbf{C}^X$ of sheaves of abelian groups on X, with values in the category of abelian groups. Then the conditions of Lemma 3.3.1 are satisfied in both of the following cases: (1) $\mathbf{M}$ is the family of flabby sheaves on X. (2) Φ is paracompactifying and $\mathbf{M}$ is the family of Φ-soft sheaves on X.*

COROLLARY. *$H^p_\Phi(X, F) = 0$ for $p > 0$ if F is flabby or if F is Φ-soft and the family Φ is paracompactifying.*

Condition (i) of the lemma has already been verified. It is trivial to verify condition (ii); only condition (iii) requires a proof, for which we go back to Godement's book (or suggest that the reader do it as an exercise).

Remark. If the family Φ is paracompactifying it is easy to verify that the fine sheaves [4, Chapter XV] are Φ-soft, and therefore show that $H^p_\Phi(X, F) = 0$. From this it follows that the cohomology theory given in [4] for paracompactifying families Φ is in fact a special case of the one developed here. We also refer to Godement's book for a particularly elegant definition of fine sheaves in terms of soft sheaves.

RESOLUTION OF THE IDENTITY. For a sheaf F, let $C^0(F)$ be the product sheaf defined by the family $F(x)$ of sets; we have a natural transformation $F \longrightarrow C^0(F)$, which moreover is injective. If we restrict ourselves to taking F in the category of abelian sheaves over X, the method in 2.5, Example (a) makes it possible to construct a resolution of the identity, $C(F)$, which is reduced to $C^0(F)$ in dimension 0, and is defined by $C^n(F) = C^0(C^{n-1}(F)/\operatorname{Im}(C^{n-2}(F)))$ in dimensions $n \geq 2$. The $C^n(A)$ are flabby and therefore Γ_Φ-acyclic whatever the cofilter Φ of closed subsets, so $\Gamma_\Phi C(A)$ is a resolving functor for Γ_Φ, and we subsequently have $H^n_\Phi(X, F) = H^n(\Gamma_\Phi C(F))$. $C(F)$ is called the *canonical resolution of F* (introduced and systematically used by Godement). If the family Φ is paracompactifying, we find another resolving functor for Γ_Φ by taking a fixed resolution C of the constant sheaf $\mathbf{Z}$ by *fine and torsion-free* sheaves and by taking for every F the complex of sheaves $F \otimes C$. What results is a resolution of F (because C is torsion free). It is an exact functor in F (same reason); moreover the $F \otimes C^n$ are also fine and therefore Γ_Φ-acyclic. Therefore the $\Gamma_\Phi(F \otimes C)$ is a resolving functor for Γ_Φ, and we subsequently have $H^n_\Phi(X, F) = H^n(\Gamma_\Phi F \otimes C)$. $F \otimes C$ will be called the Cartan resolution of F. We recall that it can be used only if Φ is paracompactifying.

AN AMUSING EXAMPLE. A space X is said to be *irreducible* if it is not the union of two proper closed subsets, that is to say if the intersection of two non-empty open subsets is non-empty; it comes to the same thing to say that every open subset of X is connected. Then every *constant* sheaf F on X is clearly flabby (the converse is also true if X is connected, as we see by taking a fibered constant sheaf not reduced to a point). In particular, *if F is a constant sheaf of abelian groups over the irreducible space X, we have $H^p(X, F) = 0$ for $p > 0$.*

3.4 Applications to questions of lifting of structure groups.[8]

Let X be an irreducible algebraic variety over an algebraically closed field k (see [15, Chapter II], whose terminology we follow), let $\mathbf{O}$ be its sheaf of local rings (= sheaf of germs of regular functions on X), and let $\mathbf{K}$ be the sheaf of germs of rational functions on X. $\mathbf{K}$ is a constant sheaf (*loc. cit.*, Proposition 9). Let $\mathbf{O}^*$ and $\mathbf{K}^*$ be, respectively, the subsheaves of $\mathbf{O}$ and $\mathbf{K}$ formed from invertible germs; clearly $\mathbf{K}^*$ is still a constant sheaf of abelian groups and $\mathbf{O}^*$ is a subsheaf of it. The quotient sheaf $\mathbf{K}^*/\mathbf{O}^* = \mathbf{D}$ is the sheaf of *germs of locally principal divisors* over X, and coincides with the sheaf of *germs of divisors* over X if the local rings $\mathbf{O}(x)$. for $x \in X$, are unique factorization domains (UFDs) (for example, if X has no singularities), which we will henceforth assume. Moreover, it can readily be seen that the sheaf $\mathbf{D}$ of germs of divisors over X is *flabby* since a section of $\mathbf{D}$ on an non-empty open set $U \subseteq X$ is a formal linear combination $\sum n_i V_i$ of irreducible hypersurfaces $V_i \subseteq U$, and is thus the restriction of the section $\sum n_i V_i$ of $\mathbf{D}$ over X. Consequently, the exact sequence $0 \longrightarrow \mathbf{O}^* \longrightarrow \mathbf{K}^* \longrightarrow \mathbf{D} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$ is a resolution of $\mathbf{O}$ by flabby sheaves ($\mathbf{K}^*$ is flabby since it is constant and X is irreducible) (cf. the end of 3.3). From this we derive the values of the $H^p(X, \mathbf{O}^*)$ (we omit the symbol Φ when we take for Φ the family of all closed subsets of X): $H^1(X, \mathbf{O}^*) = \Gamma(\mathbf{D})/\operatorname{Im}(Gamma(\mathbf{K}^*))$ = the group of the classes of divisors modulo the principal divisors (this fact is well known and results immediately from the exact cohomology sequence), and $H^i(X, \mathbf{O}^*) = 0$ if $i \geq 2$ (a result that I first obtained much more simply by the Čech cover method). We note, moreover, that this result can be extended without modification to the case where X is a "variety scheme" in the meaning of [5′], more generally in the case of "arithmetic varieties" defined by "gluing" using "spectra" of commutative rings, [8]. The application above can also be described in the framework of arithmetic varieties:

3.4.1 Proposition. *Let X be an irreducible algebraic variety (over an algebraically closed field k) whose local rings are UFDs (for example, a variety without singularities); then we have $H^i(X, \mathbf{O}^*) = 0$ for $i \geq 2$. If E is a locally trivial algebraic fibered space over X whose structure group is the projective group $GP(n-1, k)$ (cf. [20]), then E is isomorphic to the fibered space associated with a locally trivial algebraic fibered space whose structure group is $GL(n, k)$.*

For any algebraic group G, we denote by $\mathbf{O}(G)$ the sheaf of groups of germs of regular

[8] Reading this section is not necessary for understanding what follows.

functions from X to G. Then the first statement of the proposition to be proved is written $H^i(X, \mathbf{O}(k^*)) = 0$ for $i \geq 2$, and has already been proved; the second is written, using the notions and terminology developed in [11]: the canonical function

$$H^1(X, \mathbf{O}(GL(n,k))) \longrightarrow H^1(X, \mathbf{O}(GP(n-1,k)))$$

is surjective. To prove it we consider the exact sequence of algebraic groups

$$e \longrightarrow k^* \longrightarrow GL(n,k) \longrightarrow GP(n-1,k) \longrightarrow e$$

where the first homomorphism is the natural isomorphism from k^* to the center of $GL(n,k)$. We can easily see that the fibration of $GL(n,k)$ by the subgroup k^* is locally trivial (i.e there is a rational section), so the preceding exact sequence gives rise to an exact sequence of sheaves

$$e \longrightarrow \mathbf{O}(k^*) \longrightarrow \mathbf{O}(GL(n,k)) \longrightarrow \mathbf{O}(GP(n-1,k)) \longrightarrow e$$

where $\mathbf{O}(k^*)$ is in the center of $\mathbf{O}(GL(n,k))$. The proposition then results from $H^2(X, \mathbf{O}(k^*)) = 0$ and from the corollary of the following result, which generalizes [11, Proposition 5.7.2, corollary], in which we were obliged to assume paracompactness:

3.4.2 Proposition. *Let X be a topological space and $e \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow e$ be an exact sequence of sheaves of groups on X, where F is abelian. Let E be a fibered space on X with structure sheaf H, [11, Chapter IV], and let $\mathbf{F}^E$ be the sheaf of groups associated with E, and with the operations of H on F defined using the inner automorphisms of G (which operate on the invariant sheaf F). We can then define a "coboundary element" $\partial E \in H^2(X, F)$, "functorially", such that the necessary and sufficient condition for $\partial E = 0$ is that the class $c(E)$ of E in $H^1(X, H)$ [11, Chapter V], belong to the image of $H^1(X, G)$.*

This statement can be simplified when F is in the center of G, since in that case $\mathbf{F}^E = F$ no longer depends on the fiber space E and we conclude;

Corollary. *Let $e \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow e$ be an exact sequence of sheaves of groups on the space X with F in the center of G. Then we can find a ("natural") function ∂ : $H^1(X, H) \longrightarrow H^2(X, F)$ such that $\partial^{-1}(0) = \mathrm{Im}(H^1(X, G))$.*

Proof of 3.4.2.. We embed every sheaf M on X into the sheaf $\overline{M}$ whose set of sections on an open set U is the set $\prod_{x \in U} M(x)$; $\overline{M}$ is thus a *flabby* sheaf and M is identified with a subsheaf of $\overline{M}$. If M is a sheaf of groups (respectively abelian groups), the same is true of $\overline{M}$. Moreover, we can readily show that if L is a flabby sheaf of groups (not necessarily abelian) then $H^1(X, L)$ is reduced to 0 (in other words, every *principal* sheaf under L [11, Definition 3.2.4] admits a section; we can easily construct such a section by "Zornification" on the set of sections constructed on the open sets of X). From this it follows that for sheaves F of abelian groups, the group $H^1(X, F)$, as defined in [11] (by the method of

Čech), is the same as the one defined axiomatically in this work (that is, the first right satellite $S^1\Gamma$ of the functor Γ on $\mathbf{C}^X$). We can then return to the conditions of Proposition 3.4.2. We have a homomorphism of exact sequences:

$$\begin{tikzcd}
e \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & e \\
e \arrow[r] & \overline{F} \arrow[r] & \overline{G} \arrow[r] & \overline{H} \arrow[r] & e
\end{tikzcd} \qquad \overline{F} \cap G = F$$

G and F are given subsheaves of groups $\overline{G}$ and $\overline{F}$. Let $\overline{F} \cdot G$ be the sheaf of subgroups of $\overline{G}$ generated by $\overline{F}$ and G. Let

$$P = \overline{F} \cdot G \qquad\qquad F' = \overline{F}/F$$

We define in the obvious way an exact sequence of homomorphisms of sheaves of groups

$$e \longrightarrow \overline{F} \longrightarrow P \longrightarrow H \longrightarrow e$$

We show that *the corresponding function* $H^1(X,P) \longrightarrow H^1(X,H)$ *is bijective.* It is a monomorphism by virtue of the exact cohomology sequence of [11, Proposition 5.6.2], given that $H^1(X,L) = 0$ if L is locally isomorphic to F (since we easily see that sheaf that is locally isomorphic to a flabby sheaf is also flabby). It is an epimorphism, since if we let (h_{ij}) be a 1-cocycle of H relative to an open cover (U_i) of X; since $H^1(X,\overline{H}) = 0$, there exists $\overline{h}_i \in \Gamma(U_i,\overline{H})$ such that $h_{ij} = \overline{h}_i^{-1}\overline{h}_j$. Moreover, we can obviously lift the $\overline{h}_i$ to sections $\overline{g}_i(\overline{G})$ (it suffices to recall the definitions of $\overline{G}$ and $\overline{H}$). Letting $p_{ij} = \overline{g}_i^{-1}\overline{g}_j$, we can readily see that the section of H over U_{ij} defined by p_{ij} is h_{ij}, whence we conclude that $p_{ij} \in \Gamma(U_{ij},P)$, and therefore (p_{ij}) is a 1-cocycle of P defining the 1-cocycle given by passage to the quotient.

From the preceding result, we conclude that the fibered space E is isomorphic to the fibered space associated to a fibered space Q of structure space P which, moreover is well determined up to isomorphism. We note, moreover, that if we have $P = FG \subseteq \overline{G}$ operating on $\overline{G}$ by inner automorphisms, $\overline{F}$ remains stable under these operations (since $\overline{F}$ operates trivially on itself and F is invariant in G), so P operates naturally on $\overline{F}$. In addition, F is stable under the operations of P and the operations of P on F thus obtained are none other than those obtained by composing $P \longrightarrow H$ with the natural representation of H under the operations on F. From this we deduce that the associated sheaf F^E is also identified with the associated sheaf F^Q. Moreover, the operations of P also pass to the quotient $F' = \overline{F}/F$; the representations of P by germs of automorphisms of $F, \overline{F}, F'$ will be denoted by σ. We note now that there is an exact sequence of sheaf homomorphisms:

$$e \longrightarrow G \longrightarrow P \xrightarrow{u} F' \longrightarrow e$$

The homomorphism $G \longrightarrow P = \overline{F}G$ is the injection homomorphism, and therefore a homomorphism of sheaves of groups, and the homomorphism $u : P \longrightarrow F'$ is defined by making the class of $\overline{f}$ in $F'(x)$ correspond to a product $\overline{f}g$ ($\overline{f} \in \overline{F}(x)$, $g \in G(x)$; (this definition is meaningful thanks to the fact that $G \cap \overline{F} = F$). In general, this homomorphism *does not respect* multiplicative structures. but satisfies the following conditions: (i) u is surjective, and two elements of P have the same image if and only if they are congruent under right operations by G (i.e. if they define the same element of P/G); (ii) u is a crossed homomorphism[q] from P to F' (with P operating on F' as above), i.e. in each fiber $P(x)$ we have $u(e) = e$ and $u(pp') = u(p)\sigma(p)u(p')$. This situation and the datum of the fiber space Q with structure sheaf P will give rise to an element $d(Q) \in H^1(X, F'^Q)$ (where F'^Q is the sheaf associated to Q (where the relevant operations of P on F'), such that the vanishing of $d(Q)$ is necessary and sufficient for the class $c(Q) \in H^1(X,P)$ of Q to be in the image of $H^1(X,G)$. Then Proposition 3.4.2 will be proved (modulo straightforward verifications of naturality). In fact, the condition we have found is necessary and sufficient for the class $c(E) \in H^1(X,H)$ to be in the image of $H^1(X,G)$ (as we can readily see in the commutative diagram;

$$\begin{array}{ccc} H^1(X,G) & \longrightarrow & H^1(X,P) \\ \downarrow & & \downarrow \\ H^1(X,G) & \longrightarrow & H^1(X,H) \end{array}$$

where the vertical arrows are bijections). Moreover, $H^1(X, F'^Q)$ is canonically isomorphic to $H^2(X, F^Q) = H^2(X, F^E)$ since from the exact sequence $0 \longrightarrow F \longrightarrow \overline{F} \longrightarrow F' \longrightarrow 0$ we deduce the exact sequence $0 \longrightarrow F^Q \longrightarrow \overline{F}^Q \longrightarrow F'^Q \longrightarrow 0$, and $\overline{F}^Q$ is flabby because it is locally isomorphic to a flabby sheaf $\overline{F}$, then $H^i(X, \overline{F}^Q) = 0$ for $i > 0$. Then it will be sufficient to let $\partial(E) = -d(Q) \in H^2(X, F^E)$ to satisfy the desired conditions. We have to define $d(Q)$, so that it has the desired properties of functoriality and "exactness". That is what we will do under the following more general hypotheses in which the notation is changed slightly.

Let P be a sheaf of groups on X and A be a sheaf of groups with left operations from P, with the operation defined by $p \in P$ denoted by $\sigma(p)$. Let u be a crossed homomorphism from P to A, i.e. a sheaf homomorphism such that on each fibre $F(x)$, we have $u(e) = e$ and $u(pp') = u(p) \cdot \sigma(p)u(p')$. Then the subsheaf G of P, the inverse image of the null section of A under u, is a subsheaf of groups, and two elements of P have the same image in A if and only if they define the same right coset mod G. For every $x \in X$, $p \in P(X)$, and $a \in A(x)$, set

$$\rho(p)a = u(p)(\sigma(p)a)$$

[q] Translator's note: Now generally known as "derivation"

To say that u is a crossed homomorphism means precisely that the preceding formula defines a representation ρ of P by germs of automorphisms of the sheaf *of sets* A. Moreover, the homomorphism of sheaves of sets $A \times A \longrightarrow A$ defined by the product in A is compatible with the operations of P operating respectively by ρ, σ, ρ:

$$\rho(P(ab)) = (\rho(p)a)(\sigma(p)b)$$

From this we infer, for every fibered space E with structure sheaf P, a homomorphism from the sheaf of the associated sets $A(\rho)^E \times A(\sigma)^E \longrightarrow A(\rho)^E$, and it is readily seen that $A(\rho)^E$ thus becomes a sheaf of sets on which the which the sheaf of groups $A(\sigma)^E$ acts on the right and, more specifically, $A(\rho)^E$ is a *principal* sheaf (on the right) on $A(\sigma)^E$. We can consider its class $c(A(\rho)^E) \in H^1(X, A(\sigma)^E)$, which we will also denote by $d(E)$. Its vanishing is the necessary and sufficient condition for the existence of a section of the sheaf $A(\rho)^E$. We can also observe that the monomorphism $P/G \longrightarrow A$ induced by u is compatible with the operations of P, operating on P/G in the canonical fashion and on A using ρ, whence a natural monomorphism between the associated sheaves: $(P/G)^E \longrightarrow A(\rho)^E$, bijective if and only if u is surjective. Consequently the existence of a section of $(P/G)^E$, which is the necessary and sufficient condition for the class $c(E) \in H^1(X, P)$ of E to be in the image of $H^1(X, G)$, implies the existence of a section of $A(\rho)^E$, i.e. the vanishing of $d(A)$, and the converse is true if u is an epimorphism of P onto A. These considerations complete the proof of Proposition 3.4.2.

Remark 1. We put the minus sign in the formula $\partial(E) = -d(Q)$ in the proof of Proposition 3.4.2 so that we can find the usual boundary operator of the exact cohomology sequence corresponding to the exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ in the case that G (and therefore also F and H) is abelian. In that case, we have a diagram of horizontal and vertical exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{F} \arrow[r] \arrow[d] & \overline{G} \arrow[r] \arrow[d] & \overline{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & H' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whence we conclude [6, III, 4.1] that the following diagram of coboundary homomorphisms

$$\begin{tikzcd} H^0(X, H') \arrow[r] \arrow[d] & H^1(X, F') \arrow[d] \\ H^1(X, H) \arrow[r] & H^2(X, F) \end{tikzcd}$$

is *anticommutative*. The left vertical homomorphism is surjective, and we can show that in the construction above $d(Q)$ is obtained from the class $c(E) \in H^1(X, H)$ by passage to the quotient in the composite homomorphism $H^0(X, H') \longrightarrow H^1(X, F') \longrightarrow H^2(X, F)$, and therefore equals $-\partial c(E)$.

Remark 2. We should not forget that to apply to apply Proposition 3.3.1 to a projective algebraic fibered space, we must first prove that it is locally trivial. Unfortunately, this proof, when we do not know a priori that we can lift the structure group, can seem difficult. Example: On a complex projective variety without singularities, we have a *holomorphic* fibred space E whose fibre is a projective space. We know from Kodaira-Borel that E is also an algebraic variety; does it come from a holomorphic fibered vector space? The response is affirmative in accordance with Proposition 3.4.1 if E is locally trivial from an algebraic standpoint, i.e., every point of the base has a rational section passing through it; moreover the converse is true because, according to [16], every holomorphic fibered vector space on X is algebraically locally trivial.

Remark 3. The situation $u : P \longrightarrow A$ (where u is a crossed homomorphism of sheaves) described above is encountered in a variety of interesting situations, for example: X is a holomorphic variety, G is a complex Lie group having the Lie algebra V, P is the sheaf of germs germs of holomorphic functions from X to G, A is the sheaf of holomorphic differential 1-forms on X with values in V, on which P acts via the adjoint representation, and we set $u(g) = (dg)g^{-1}$. This allows us to associate to each holomorphic fibered space E on X with structure group G a class $d(E) \in H^1(X, \Omega^1(\mathrm{ad}(E)))$, where $\mathrm{ad}(E)$ denotes the "adjoint" fibered vector space (with fibre V) of E, and $\Omega^1(\mathrm{ad}(E))$ denotes the sheaf of germs of holomorphic differential 1-forms with values in $\mathrm{ad}(E)$. Since the kernel of u is the subsheaf of P formed from the germs of constant functions from X to G, we see that the vanishing of $d(E)$ is a necessary condition for the structure sheaf of E to be reducible to the *constant* sheaf G, i.e., for the existence of an integrable holomorphic connection; and this condition is sufficient when X has complex dimension 1 (since then u surjective). The invariant $d(E)$ was first introduced by Weil [21]; a more geometric definition by Atiyah [1] proves that the vanishing of $d(E)$ is in all cases necessary and sufficient for the existence of a holomorphic connection (not necessarily integrable) on E.

3.5 The exact sequence of a closed subspace

Let Y be a locally closed subspace (i.e. the intersection of an open subset and closed subset) of the space X. For each abelian sheaf F over Y, there exists a unique abelian sheaf over X whose to restriction to Y is F and whose restriction to $\complement Y$ is 0. To see this, we are immediately led to the case in which Y is either open or closed for which the simple proof is given in [4, XVII, Proposition 1]. This sheaf over X will be denoted F^X. $F \longrightarrow F^X$ is an exact functor $\mathbf{C}^Y \longrightarrow \mathbf{C}^X$; moreover, if $Z \subseteq Y \subseteq X$ (Z locally closed in Y and hence in X), and if F is an abelian sheaf over Z, then $(F^Y)^X = F^X$. Now if F is an abelian sheaf over X, we set $F_Y = (F|Y)^X$; this is the sheaf over X characterized by the condition that its restriction to Y is the same as that of F, while the restriction to $\complement Y$ is 0. F_Y is an exact functor $\mathbf{C}^X \longrightarrow \mathbf{C}^X$; moreover, we have the transitivity property $(F_Y)_Z = F_Z$ if $Z \subseteq Y \subseteq X$ as above. If we assume that Y is closed and that therefore $U = \complement Y$ is open, we have a well-known exact sequence

$$0 \longrightarrow F_U \longrightarrow F \longrightarrow F_Y \longrightarrow 0$$

for every abelian sheaf F over X. Recall that we have $\Gamma(F_Y) = \Gamma(Y, F) = \Gamma(F|Y)$, while $\Gamma(F_U)$ is identified with the subgroup of $\Gamma(F)$ consisting of sections whose support is contained U. Let Φ be a cofilter of closed subsets of X. For every subset Z of X, let Φ_Z be the "induced" cofilter consisting of the $A \in \Phi$ that are contained in Z (not to be confused with the trace $\Phi \cap Z$ of Φ over Z). We can readily see that if Z is locally closed and Φ is paracompactifying (cf. 3.3), then Φ_Z is also paracompatifying. If Z is locally closed, Φ arbitrary, we can easily deduce from the formulas above that the following more general formula (which holds. in particular, when Z is open or closed):

$$\Gamma_\Phi(X, F_Z) = \Gamma_{\Phi_Z}(Z, F|Z)$$

which is valid for any abelian sheaf over X. Moreover, this is equivalent to the following:

$$\Gamma_{\Phi_Z}(Z, G) = \Gamma_\Phi(X, G^X)$$

which is valid for any abelian sheaf over Z. Since G^X is an *exact* functor in G, the $H^p_\Phi(X, G^X)$ form a cohomological functor on $\mathbf{C}^Z$ and, since the universal cohomological functor $(H^p_{\Phi_Z}(Z, G))$ coincides with the first in dimension 0, we can derive *canonical homomorphisms*

$$H^p_{\Phi_Z}(Z, G) \longrightarrow H^p_Z(X, G^X)$$

(characterized by the definition of homomorphism of cohomological functors that reduces in dimension 0 to the one above); or, starting from a sheaf F over X:

$$H^p_{\Phi_Z}(Z, F|Z) \longrightarrow H^p_\Phi(X, F_Z)$$

3.5.1 THEOREM. *The preceding homomorphisms are isomorphisms in each of the two cases below:*

01. *Z is closed;*

02. *Φ is paracompactifying and Z is open.*

PROOF. It is sufficient in each case to show that the functors $H^p_\Phi(X, G^X)$ on $\mathbf{C}^Z$ are effaceable for $p \geq 1$. If Z is closed, that results from the fact that when G is injective, G^X is injective (Proposition 3.1.4). If Z is open and if Φ is paracompactifying, the same is true for Φ_Z; therefore every $G \in \mathbf{C}^Z$ is embedded in a Φ_Z-soft sheaf (cf. 3.3). It is then sufficient to note that if G is Φ_Z-soft, then G^X is Φ-soft (a fact that is readily proved), and to apply the corollary to Proposition 3.3.2. ∎

To continue, the exact sequence $0 \longrightarrow F_U \longrightarrow F \longrightarrow F_Y \longrightarrow 0$ corresponding to a closed subspace Y and its open complement U gives rise to an exact sequence of cohomology which can be written, thanks to the preceding theorem:

$$\cdots \longrightarrow H^P_\Phi(X, F_U) \longrightarrow H^p_\Phi(X, F) \longrightarrow H^p_{\Phi_Y}(Y, F) \longrightarrow H^{p+1}_\Phi(X, F_U) \longrightarrow \cdots$$

(where, to simplify, we have written F in place of $F|Y$ in the third term). If Φ is paracompactifying, we can in addition replace terms of the form $H^p(X, F_U)$ by $H^p_{\Phi_U}(U, F)$, and we then obtain the well-known exact sequence from [4, XVII].

3.6 On the cohomological dimension of certain spaces

3.6.1 PROPOSITION. *Let X be a topological space and (T^p) be a covariant cohomological functor defined on the category $\mathbf{C}^X$ of sheaves of abelian groups on X with values in a category $\mathbf{C}'$. We assume that $\mathbf{C}'$ satisfies condition* AB 4) *(cf. 1.5), which implies that we can form inductive limits in $\mathbf{C}'$ (Proposition 1.8), and we assume that the T^p commute with inductive limits. Let $F \in \mathbf{C}^X$; then $T^p(F)$ belongs to any thick subcategory (cf. 1.11) of $\mathbf{C}''$ of $\mathbf{C}'$ stable under infinite direct sums, in which all the objects of the form $T^i(\mathbf{Z}_U)$, where U is an arbitrary open set in X and $i = p$, $p+1$, or $p+2$.*

(The meaning of $\mathbf{Z}_U$ is the same as in the preceding section.) Consider a family $(f_i)_{i\in I}$ of sections of F over open sets U_i. Each f_i defines a homomorphism from $\mathbf{Z}_{U_i}$ to F, so (f_i) defines a homomorphism from the direct sum $\bigoplus_i \mathbf{Z}_{U_i}$ to F. We say that (f_i) is a *system of generators* of F if the preceding homomorphism is an epimorphism. It is trivial to show that a family of generators exists for any F, from which it immediately follows that F is an inductive colimit of an increasing filtered family of subsheaves F_j, each of which admits a *finite* family of generators. Since $\mathbf{C}''$ is thick and stable under infinite direct sums, it is also stable under inductive limits (because an inductive limit of objects in $\mathbf{C}'$ is isomorphic to a quotient of their direct sum). Since $T^p(F) = \varinjlim_j T_p(F_j)$, to prove $T^p(F) \in \mathbf{C}''$, it suffices to show that $T^p(F_j) \in \mathbf{C}''$ for every j, which reduces to the case that F admits a *finite* family of generators $(f_i)_{1\leq i\leq k}$. We denote by F_n $(0 \leq n \leq k)$ the subsheaf of F generated by the F_i with $1 \leq i \leq n$. The F_n form a finite increasing sequence of subsheaves of F whose successive quotients F_n/F_{n-1} $(1 \leq n \leq k)$ each admit a *single* generator. Reasoning by recursion on the length k of the sequence, and using the fact that T_p is half exact, we can prove that $T^p(F_n/F_{n-1}) \in \mathbf{C}''$ for every n. This reduces to the case that F is generated by a *single* generator, that is, when there is an exact sequence of sheaves

$$0 \longrightarrow R \longrightarrow \mathbf{Z}_U \longrightarrow F \longrightarrow 0$$

where R is a subsheaf of $\mathbf{Z}_U$ and therefore also of $\mathbf{Z}$. From this we get the exact sequence $T^p(\mathbf{Z}_U) \longrightarrow T^p(F) \longrightarrow T^{p+1}(R)$ and since we assume $T^p(\mathbf{Z}_U) \in \mathbf{C}''$, we can prove that $T^{p+1}(R) \in \mathbf{C}''$. Thus the subsheaf R of the constant sheaf $\mathbf{Z}$ is generated by the family $(f_i)_{i\in I}$ of *constant* sections n_i of $\mathbf{Z}$ over open sets U_i. Proceeding as above, we reduce to the case that the family is finite. We can assume, of course, that the $n_i < 0$. Moreover, we can assume that the f_i are chosen so that for every $x \in X$, there exists, among those n_i for which U_i contains x, a generator of the subgroup $R(x)$ of the group $\mathbf{Z}(x) = \mathbf{Z}$ of integers. To do so for any subset $(i_1, ldots, i_p)$ of the set $[1, k]$ of the first k positive integers, it suffices to consider the section $f_{i_1,\dots,i_p}$ of F on $U_{i_1} \cap \dots \cap U_{i_p}$ whose value is the gcd of $n_{i_1}, \dots, n_{i_p}$ and to adjoin these sections to the system of generators. We thus conclude that the assumption about the f_i is satisfied. We assume $n_1 \leq n_2 \leq \dots \leq n_k$. If we denote by R_m $(0 \leq m \leq k)$ the subsheaf of R generated by the sections f_i with $1 \leq i \leq m$, the R_m form a finite increasing sequence of subsheaves of R, and we again show, as above, by recursion on k, and by using the fact that T^{p+1} is half exact, that to prove $T^{p-1}(R) \in \mathbf{C}''$, it suffices to prove that $T^{p+1}(F_m/F_{m-1}) \in \mathbf{C}''$ for $1 \leq m \leq k$. Now for any m, let V_m be the union of the U_i for $1 \leq i \leq m$, and let $Y_m = U_m - V_{m-1}$. I claim that F_m/F_{m-1} *is isomorphism to* $\mathbf{Z}_{Y_m}$. In fact, its restriction to $\complement V_m$ is obviously 0 (since $\complement F_m$ is already 0), and the same is true for its restriction to V_{m-1}. It is sufficient to prove it for any $x \in V_{m-1} \cap U_m$. By assumption, among the n_i corresponding to the $U_i \subseteq X$, is included their gcd n_{i_0} which must therefore divide both n_m and an n_i with $i < m$ (take an index $i < m$ with $x \in U_i$, which exists because $x \in V_{m-1}$). Then either $i_0 < m$, which proves that $f_{n_m}(x) \in F_{m-1}(x)$, or $i_0 \geq m$, whence $n_{i_0} \geq n_m$ and thus $n_{i_0} = n_m$, therefore n_m (dividing n_i and $\geq n_i$) is equal to n_i, from which $f_{n_m}(x) \in F_{m-1}(x)$, so in both cases $F_{m-1}(x) = F_m(x)$. Thus the restriction of F_m/F_{m-1} to $\complement V_m \cup V_{m-1}$ i.e., to $\complement Y_m$, is 0. Moreover, the restriction of F_m/F_{m-1} to $Y_m = U_m - V_{m-1}$ is isomorphic to the restriction of F_m (since the restriction of F_{m-1} is 0) and is generated by the restriction of the section f_m and therefore is isomorphic to the constant sheaf $\mathbf{Z}$. This proves that F_m/F_{m-1} is isomorphic to $\mathbf{Z}_{Y_m}$, so we can prove that $T^{p+1}(\mathbf{Z}_Y) \in \mathbf{C}''$ if Y is a locally closed subset of X. Then we have $Y = V - U$, where U and V are two open subsets of X. Let $W = U \cup V$ which is open in X. We have an obvious exact sequence: $0 \longrightarrow \mathbf{Z}_U \longrightarrow \mathbf{Z}_W \longrightarrow \mathbf{Z}_Y \longrightarrow 0$, whence an exact sequence $T^{p+1}(\mathbf{Z}_W) \longrightarrow T^{p+1}(\mathbf{Z}_Y) \longrightarrow T^{p+2}(\mathbf{Z}_U)$. By assumption, the extreme terms of this exact sequence are in $\mathbf{C}''$; the same is therefore true of $T^{p+1}(\mathbf{Z}_Y)$, which completes the proof of the proposition. ∎

3.6.2 PROPOSITION. *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be two abelian categories satisfying Axiom* AB 5) *(cf. 1.5). We assume that* $\mathbf{C}$ *admits a generator (cf. 1.9). Let* T *be a covariant functor from* $\mathbf{C}$ *to* $\mathbf{C}'$. *For the right derived functors* R^pT *to commute with inductive limits, it suffices that:*

(a) T *commutes with the inductive limits;*

(b) *If* $M = \varinjlim_i M_i \in \mathbf{C}$ *where the* M_i *are injective, then* M *is* T*-acyclic, i.e.* $R^pT(M) = 0$ *for* $p > 0$.

(Condition (b) is clearly necessary for the conclusion to hold and (a) is necessary if T is left exact, and thus $R^0T = T$.) We note first that the R^pT are well defined by virtue of Theorem 1.10.1. Let $(A_i)_{i\in I}$ be an inductive system in $\mathbf{C}$ and A its inductive limit. We wish to prove that the natural morphisms $\varinjlim R^pT(A_i) \longrightarrow R^pT(A)$ are bijective. We will first show that *there exists an inductive system* $(C_i)_{i\in I}$ *of complexes* (with values in $\mathbf{C}$) *and an "augmentation"* $(A_i) \longrightarrow (C_i)$ *such that for every* $i \in I$, $A_i \longrightarrow C_i$ *is a right resolution of* A *by injectives.* We consider the category $I(\mathbf{C})$ of inductive systems on I with values in $\mathbf{C}$. It is a category of diagrams, which according to Proposition 1.6.1 satisfies the same assumptions as $\mathbf{C}$. According to Theorem 1.10.1, every object in this category thus admits a right resolution by injectives. We can readily show that if (M_i) is an injective object of $I(\mathbf{C})$, then the M_i are injectives in $\mathbf{C}$ (this is a general fact for categories of diagrams for a scheme Σ satisfying the general conditions of Proposition 1.9.2.) Next we consider a right resolution of (A_i) by injectives of $I(\mathbf{C})$, $0 \longrightarrow (A_i) \longrightarrow (C_i^0) \longrightarrow (C_i^1) \longrightarrow \cdots$. For every i, let C_i be the complex $0 \longrightarrow C_i^0 \longrightarrow C_i^1 \longrightarrow \cdots$. Then we see that the system C_i answers the question. Since the functor $\varinjlim$ on $I(\mathbf{C})$ is exact (Proposition 1.8), we obtain a resolution C of A by the $C^p = \varinjlim_i C^{p_i}$. Under Condition (b), the C^p are T-acyclic, therefore we have $R^pT(A) = H^p(T(C))$. Since T commutes with inductive limits, by virtue of (a), we have $T(C) = \varinjlim T(C_i)$. Thus by virtue of the exactness of the functor $\varinjlim$ on $I(\mathbf{C}')$ (resulting from Axiom AB 5) using Proposition 1.8), we obtain $H^p(T(C)) = \varinjlim H^p(T(C_i))$. Since C_i is the complex associated with an injective resolution of A_i, we have $H^pT(C_i) = R^pT(A_i)$, whence the desired conclusion $R^pT(A) = \varinjlim R^pT(A_i)$.

3.6.3 Proposition. *Let X be a topological space equipped with a cofilter Φ of closed subsets. Then the functors $H^p_\Phi(X,F)$ on $\mathbf{C}^X$ commute with the inductive limit in the following two cases:*

1. *X is locally compact and Φ is the family of compact subsets;*

2. *X is a Zariski space and Φ is the family of all its closed subsets.*

(We call a *Zariski space* a space in which every decreasing sequence of closed subsets is stationary (cf. [15, Page 223]).) It is sufficient to verify Conditions (a) and (b) of Proposition 3.6.2 for the functor Γ_Φ. Verifying (a) is an easy exercise using compactness and is left to the reader (see also [9]). Condition (b) will result from the corollary to Proposition 3.3.1 and from Lemma 3.6.4.

3.6.4 Lemma. *In case 1 every inductive limit of Φ-soft sheaves is Φ-soft. In case 2, every inductive limit of flabby sheaves is flabby.*

Let us assume Condition 1. Let (F_i) be an inductive system of soft sheaves on X, F the inductive limit, and f a section of F on an $A \in \Phi$, i.e. on a compact subset A. If we apply Proposition 3.6.2 (a) to the compact set A, f comes from a section f_i of an F_i over A, and since F_i is Φ-soft, this f_i is the restriction of a $g_i \in \Gamma_\Phi(F_i)$, therefore f is the restriction to A of the $g \in \Gamma_\Phi(F)$ defined by g_i. Case 2 works analogously by noting that a subset U of a Zariski space is a Zariski space, therefore (a) applies to it.

We will say that a space X is of *cohomological dimension* $\leq n$ if $H^i(X,F) = 0$ for $i > n$ for every abelian sheaf F on X. Combining Propositions 3.6.1 and 3.6.3, we find

Corollary. *Let X be a space that is either compact or a Zariski space. Let n be a non-negative integer. For X to be of cohomological dimension $\leq n$, it is necessary and sufficient that $H^i(X, \mathbf{Z}_U) = 0$ for $i > n$ and for every open set $U \subseteq X$.*

We are coming to the essential results of this section. Let X be a Zariski space. We say that X is of *combinatorial dimension* $\leq n$ if every strictly decreasing sequence of *irreducible* closed subsets has length at most $n+1$. That said:

3.6.5 Theorem. *Let X be a Zariski space of combinatorial dimension $\leq n$. Then X is of cohomological dimension $\leq n$, i.e. we have $H^i(X,F) = 0$ for $i > n$ and for every abelian sheaf F on X.*

We argue by induction on the combinatorial dimension n of X; the theorem is trivial if $n = 0$ (then X is a discrete finite set). Suppose this has been proven for spaces of combinatorial dimension $< n$, where $n \geq 1$, and we will prove it if X has combinatorial dimension $\leq n$. Let X_k range over the irreducible components of X [15, Chapter 2, Proposition 2]. If F is an abelian sheaf on X, we have a natural monomorphism from F to the direct sum of the $F_k = F_{X_k}$, whence an exact sequence

$$0 \longrightarrow F \longrightarrow \bigoplus F_k \longrightarrow R \longrightarrow 0$$

where R is a sheaf whose support is contained in $Y = \bigcup_{k \neq l}(X_k \cap X_l)$, which is of combinatorial dimension $\leq n-1$. From this we derive an exact sequence

$$H^{i-1}(X, R) \longrightarrow H^i(X, F) \longrightarrow \bigoplus_k H^i(X, F_k)$$

If $i > n$ (whence $i-1 > n-1$), we have $H^{i-1}(X, R) = H^{i-1}(Y, R) = 0$ by induction. To prove that $H^i(X, F) = 0$, it therefore suffices to prove that $H^i(X, F_k) = 0$. Then we have $H^i(X, F_k) = H^i(X_k, F_k)$, from which we are reduced to proving the theorem for the irreducible space X_k. We therefore assume that X is irreducible. From the corollary to Theorem 3.6.3, it is sufficient to prove that $H^i(X, \mathbf{Z}_U) = 0$ for $i > n$ and for every open subset U of X. We can assume $U \neq \emptyset$ so $Y = \complement U$ is a proper closed subset of X, thus of combinatorial dimension $\leq n-1$. From the exact sequence $0 \longrightarrow \mathbf{Z}_U \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}_Y \longrightarrow 0$ we get, since $\mathbf{Z}$ is flabby (cf. end of 3.3), $H^i(X, \mathbf{Z}_U) = H^{i-1}(X, \mathbf{Z}_Y) = H^{i-1}(Y, \mathbf{Z})$ which is 0 by induction.

Remark 1. The preceding theorem generalizes an earlier theorem of Serre [18].

Remark 2. We can find Zariski spaces of cohomological dimension 0 and of arbitrarily large finite or of infinite combinatorial dimension. It is sufficient to consider on a finite or infinite well-ordered set X the topology whose closed sets are sets of the form X_x, where for every $x \in X$, X_x denotes the set of $y \in X$ such that $y < x$.

3.7 The Leray spectral sequence of a continuous function

Let f be a continuous function from a space Y to a space X. We assume given in Y an cofilter Ψ of closed subsets. For every open subset $U \subseteq X$, let $\Psi(U)$ be the cofilter of closed subsets in $f^{-1}(U)$ formed from the $A \subseteq f^{-1}(U)$ such that for every $x \in U$ there is a neighborhood $V \subseteq U$ of x such that the closure[r] of $A \cap f^{-1}(V)$ belongs to Ψ. In particular, $\Psi(X)$ denotes a cofilter of closed subsets of Y. Let F be a sheaf of abelian groups on Y. For every open $U \subseteq X$, we consider the group $\Gamma_{\Psi(U)}(f^{-1}(U), F)$. It is easy to verify that for the functions with obvious restrictions (relative to inclusions $V \subseteq U$), these groups form a *sheaf* on X, denoted $f(F)$ and called the *direct image of F by f relative to* Ψ. If Ψ is the family of all closed subsets of Y, we simply write $f_*(F)$ instead of $f_\Psi(F)$, and $f_*(F)$ is called the *direct image of F by f*. Moreover, in this case, $\Psi(U)$ is the set of all the closed

[r]Translator's note: The French word was “adhérence”, not “fermeture”.

subsets of the space $f^{-1}(U)$, therefore $\Gamma(U, f_*(F)) = \Gamma(f^{-1}(U), F)$ (which also makes sense even if we assume only that F is a sheaf of sets). By definition we have, in the general case:

$$\Gamma(U, f(F)) = \Gamma_{\Psi(U)}(f^{-1}(U), F)$$

We readily verify the formula

$$\operatorname{supp}_X(\phi) = \operatorname{cl}(f(\operatorname{supp}_Y \phi))$$

for every $\phi \in \Gamma(f_\Psi(F)) = \Gamma_{\Psi(X)}(F)$ (for which we consider the supports in X and in Y, which we distinguish in the notation by putting the name of the space as a subscript to the supp). Now let Φ be an cofilter of closed subsets in X. We denote by Ψ' the cofilter of those closed subsets A of Y which are in $\Psi(X)$ and are such that the closure of $f(A)$ is in Φ. Then the preceding formulas imply the formula:

$$\Gamma_\Phi(f_\Psi(F)) = \Gamma_{\Psi'}(F)$$

(The families Φ and Ψ are called *adapted (relative to f)* if we have $\Psi = \Psi'$. This is the case, for example, if Φ and Ψ consist of all the closed subsets of X, respectively Y.)

f_Ψ *is a left exact functor* from the category $\mathbf{C}^Y$ of abelian sheaves on Y to the category $\mathbf{C}^X$; moreover, the preceding formula denotes a *natural equivalence*. We can write it

$$\Gamma_{\Psi'} = \Gamma_\Phi f_\Psi$$

We wish to apply Theorem 2.4.1 to this. To do so, it is necessary to give the conditions by which f_Ψ transforms an injective sheaf into a Γ_Φ-acyclic sheaf.

3.7.1 LEMMA. *1. If Ψ is the set of all closed subsets of Y, f_Ψ transforms injective sheaves into injective sheaves.*

2. If Φ is paracompactifying, f_Ψ transforms flabby sheaves into Φ-soft sheaves.

PROOF. 1. We assume that F is injective. For every $y \in Y$, let $M(y)$ be an injective abelian group containing $F(y)$ and let M be the product sheaf defined by the $M(x)$ (cf. 3.1). F is a subsheaf of M and thus a direct factor of M since it is injective, therefore $f_\Psi(F)$ is a direct factor of $f_\Psi(M)$, and it is sufficient to prove that $f_\Psi(M)$ is injective. For every $x \in X$, let $N(x)$ be the product of the $M(y)$ for $y \in f^{-1}(x)$. It follows by definition that $f(M)$ is the product sheaf N defined by the family of $N(x)$. Each $N(x)$ is an injective abelian group, being a product of injective groups, therefore N is injective (Proposition 3.1.2) and therefore $f(M)$ is injective.

2. Assume that F is flabby. We will prove that $f_\Psi(F)$ is Φ-soft. Let g be a section of $f_\Psi(F)$ over $B \in \Phi$. We are looking for an $h \in \Gamma_\Phi(f_\Psi(F)) = \Gamma_{\Psi'}(F)$ [s] whose restriction to

[s] Translator's note: The first character on the subscript to Γ was missing, but Ψ' appears later and we are guessing that this formula defines it.

B is g. Since Φ is paracompactifying, B has a paracompact neighborhood $B' \in \Phi$, thus g is the restriction of a section g' of $f_\Psi(F)$ defined on an appropriate neighborhood $U \subseteq B'$ of B. Since B' is normal, there is a *closed* neighborhood B_1 of B contained in U; let U_1 be its interior. We consider g' to be an element of $\Gamma_{\Psi(U)}(f^{-1}(U), F)$. Its support A is a closed subset of $f^{-1}(U)$, therefore $A \cap f^{-1}(B_1)$ is a closed subset of Y, and therefore its complement in Y is open. Since the intersection of this open set with the open set $f^{-1}(U_1)$ is contained in $\complement A$, $g'y = 0$, there is a section g_1 of f on the open set that is the union of $f^{-1}(U_1)$ and $\complement(A \cap f^{-1}(B_1))$ which coincides with g' on the first and vanishes on the second. Finally, since F is flabby there is a section h of F on Y that induces g_1. The support of h is contained in $A \cap f^{-1}(B_1)$, whence it follows immediately that it is Ψ'. Therefore h can be considered to be an element of $\Gamma_\Phi(f_\Psi(F))$, obviously inducing on U_1 the same section as g', and consequently inducing g on B. ■

It follows from the corollary to Proposition 3.3.1 that in each of the conditions of Lemma 3.7.1, we can apply Theorem 2.4.1 to the composite functor $\Gamma_{\Psi'} = \Gamma_\Phi f_\Psi$: there is a cohomological spectral functor on $\mathbf{C}_Y$ converging to the graded functor $(H^n_{\Psi'}(Y, F))$, whose initial term is

$$E_2^{p,q} = H^p_\Phi(X, (R^q f_\Psi)(F))$$

It remains to make the sheaves $(R^q f_\Psi)(F)$ explicit. In general:

3.7.2 Lemma. *Let T be a covariant functor from an abelian category $\mathbf{C}$ to the category $\mathbf{C}^X$ of abelian sheaves on X. We assume that every object of $\mathbf{C}$ is isomorphic to a subobject of an injective such that the right derived functors R^qT exist. Then for every $A \in \mathbf{C}$, the sheaf $R^qT(A)$ can be identified with the sheaf associated (cf. 3.1) with the presheaf that associates to every open set $U \subseteq X$, $R^q(\Gamma^U T)(A)$ (where Γ^U denotes the functor $F \mapsto \Gamma(U, F)$ on $\mathbf{C}^X$).*

Let $C(A)$ be the complex associated to a right resolution of A by injectives; we thus have $R^qT(A) = H^q(T(C(A)))$. The qth cohomology sheaf of the complex of sheaves $K = (K^i)$ is none other than the sheaf associated to the presheaf that associates the group $H^i(\Gamma(U, K))$ to the open set U. Thus $R^qT(A)$ is the sheaf associated to the presheaf $U \mapsto H^i(\Gamma(U, T(C(A)))) = R^q(\Gamma^U T)(A)$, which proves the lemma.

In the present case $T = f_\Psi$, we see that $R^q f_\Psi(F)$ is the sheaf associated to the presheaf $U \mapsto R^q(\Gamma^U f_\Psi)(F) = R^q\Gamma_{\Psi(U)}(F)$. We have already noted, as an immediate consequence of Proposition 3.1.3, that the derived functors of $\Gamma_{\Psi(U)}(f^{-1}(U), F)$ are none other than the $H^q_{\Psi(U)}(f^{-1}(U), f)$. We thus obtain:

3.7.3 Theorem. *Let f be a continuous function from a space Y to a space X. We assume that X and Y are equipped with cofilters Φ and Ψ, respectively, of closed subsets. The notation $\Psi(U)$, f_Ψ, and Ψ' are the same as at the beginning of this section. We assume that Φ is paracompactifying or that Ψ is the set of all closed subsets of Y. Then there is a* *cohomological spectral functor on the category* $\mathbf{C}^Y$ *of abelian sheaves* F *over* Y*, converging to the graded functor* $(H^n_{\Psi'}(Y,F))$ *whose initial terms is*

$$E_2^{p,q}(F) = H^p_\Phi(X, R^q f_\Psi(f))$$

In this formula, $R^q f_\Psi(F)$ *is the sheaf over* X *associated to the presheaf that associates the group* $H^q_{\Psi(U)}(f^{-1}(U)F)$ *to the open set* U *in* X*.*

The simplest case is the one in which Φ and Ψ are the sets of all closed subsets of X and Y, respectively. Then, *without any assumptions about* X*,* Y*, or* f*, we find a spectral functor converging to* $(H^n(Y,f))$ *whose initial term is* $H^p(X, R^q f(F))$*, where* $R^q(f)(F)$ *is the sheaf associated to the presheaf* $U \mapsto H^q(f^{-1}(U), F)$*.* This statement can be usefully applied, for example, in the cohomology theory of algebraic varieties (equipped with their Zariski topology).

We will limit ourselves here to this statement of the natural validity conditions of the Leray spectral sequence, which we will not study any further.

3.8 Comparison with Čech cohomology

We refer to [15] for the definition of "cohomology groups" of X with coefficients in an abelian sheaf F, calculated using the method of Čech covers. We will denote these groups by $\check{H}^p(X,F)$ to distinguish them from the groups $H^p(X,F)$ defined in Section 2. (To simplify, we will not consider any "Φ-family" other than the set of *all* closed sets.) We note, however, that these groups can be defined by assuming only that F is a *presheaf* of abelian groups: for every open cover $\mathbf{U} = (U_i)$ of X, we can form the complex $C(\mathbf{U},F) = \sum_p C^p(\mathbf{U},F)$ of the cochains of $\mathbf{U}$ with values in the presheaf F and we can set $H^p(\mathbf{U},F) = H^p(C(\mathbf{U},F))$, and then take

$$\check{H}^p(X,F) = \varinjlim H^p(\mathbf{U},F)$$

with the inductive limit taken over the filtered partial order of all "classes of open covers" of X (two open covers being considered equivalent if each refines the other).

Unfortunately, the $\check{H}^p(X,F)$ do not in general form a cohomological functor on the category $\mathbf{C}^X$ of sheaves of abelian groups over X (see the example at the end of this section). But $(\check{H}^0, \check{H}^1)$ forms an exact ∂-functor, [15, 11]. In addition, the $\check{H}^p$ are *effaceable* functors for $p > 0$: for this, it is sufficient to show, for example, that if M is the product sheaf defined by a family $(M_x)_{x\in X}$ of abelian groups (cf. Section 1), then the $\check{H}^p(X,M)$ vanish for $p > 0$. In fact, we even prove $H^p(\mathbf{U},M) = 0$ for every open cover $\mathbf{U}$ of X, using the well-known homotopy operator, employed classically in the case that M is fine and X is paracompact. It follows that, *if* X *is such that the* $\check{H}^p$ *can be considered as the components of a cohomological functor, then the* $\check{H}^p$ *are canonically isomorphic to the functors* H^p*. This* *is true if X is paracompact* (cf., for example [15, Section 25]) or if X arbitrary, but then it holds only for $p = 0$ and $p = 1$ (as we have already remarked in Section 4).

More specific results are connected to the Cartan-Leray spectral sequence of a cover, as was pointed out to me by H. Cartan (whose idea I am borrowing). Let $\mathbf{U}$ be a fixed open cover of X. We set $C(F) = C(\mathbf{U}, F)$ for every sheaf F. We thus obtain a left exact covariant functor from $\mathbf{C} = \mathbf{C}^X$ to the category $\mathbf{C}'$ of complexes of abelian groups of non-negative degree. Moreover, we have seen that the $H^p(C(F))$ are effaceable. We easily derive from this a spectral functor converging to the right derived functor $R(H^0C)$ of the functor H^0C whose initial term is $E_2^{p,q} = H^p(R^qC)$. We can see this either directly by taking an injective resolution of $F \in \mathbf{C}$ and looking at the spectral sequences of the bicomplex obtained by transforming this resolution by $\mathbf{C}$ or, even better, by remarking that if we consider $H^0(K)$ to be a left exact covariant functor on $\mathbf{C}'$ (with values in the category of abelian groups), its right derived functors are the $H^p(K)$, so that our spectral sequence is simply a special case of Theorem 2.4.1. Of course, $(R^qC)(F)$ is the complex whose components are the $(R^qC^p)(F)$ if the C^p are the components of C. In the case we are interested in, the R^pC^q remain to be specified. By virtue of

$$C^pF = \prod_{\sigma^p} \Gamma(U_{\sigma^p}, F)$$

(with the product extended to all sequences $\sigma^p = (i_0, \ldots, i_p)$ of the $p+1$ indices of the cover $U = (U_i)_{i \in I}$), we readily see that

$$R^qC^p(F) = \prod_{\sigma^p} R^q(\Gamma(U_{\sigma^p}, F))$$

If V is an open subset of X, the right derived functors of the functor $\Gamma(V, F) = \Gamma(F|V)$ can be easily specified thanks to the fact that the restriction functor $F \mapsto F|V$ from $\mathbf{C}^X$ to $\mathbf{C}^V$ is exact and transforms injectives into injectives (Proposition 3.1.3): we will have $R^q(\Gamma(V, F)) = (R^q\Gamma^V)(V|F) = H^q(V, F)$. Thus we can denote by $H^q(F)$ the *presheaf* on X whose value on an open set V is $H^q(V, F)$. We then have

$$R^qC^p(F) = \prod_{\sigma^p} \Gamma(U_{\sigma^p}, H^q(F)) = C^p(\mathbf{U}, H^q(F))$$

Of course, the boundary operator on $R^qC(F) = \Sigma_p R^qC^p(F)$ is the one on $C(\mathbf{U}, H^q(F))$, from which we finally have $E_2^{p,q}(F) = H^p(\mathbf{U}, H^q(F))$. As for the convergence of the spectral sequence, it is the right derived functor of $H^0C(F) = \Gamma(X, F)$, i.e. the functor $(H^n(X, F))$ whence:

3.8.1 Theorem. *Let X be a topological space equipped with an open cover $\mathbf{U}$. Then there exists a cohomological spectral functor on the category $\mathbf{C}^X$ of sheaves of abelian groups over*

X, converging to the graded functor $(H^n(X,F))$ whose initial term is

$$E_2^{p,q} = H^p(\mathbf{U}, H^q(F))$$

where $H^q(F)$ denotes (for every sheaf $F \in \mathbf{C}^X$) the presheaf $V \mapsto H^q(V,F)$ over X.

We will note that this spectral sequence is constructed without any hypothesis of paracompactness of X or local finiteness on $\mathbf{U}$. The preceding spectral sequence gives natural transformations

$$H^p(\mathbf{U}, F) \longrightarrow H^p(X, F)$$

and also:

Corollary 1. *The preceding natural transformations are equivalences if all the $U_{i_0,\ldots,i_p}$*[t] *are F-acyclic (that is they satisfy $H^p(U_{i_0,\ldots,i_p}, F) = 0$ for $p > 0$).*

We restrict ourselves now to the cover $\mathbf{U} = (U_x)_{x\in X}$ indexed by the points $x \in X$ such that $x \in U_x$ for every $x \in X$; we order them by writing $\mathbf{U} \leq \mathbf{U}'$ if $U_x \subseteq U'_x$ for all x. If $C_{\mathbf{U}}$ and $C_{\mathbf{U}'}$ are the corresponding functor-complexes on $\mathbf{C}^X$, we have a natural transformation $C_{\mathbf{U}} \longrightarrow C_{\mathbf{U}'}$ from which we get a natural transformation between the corresponding spectral functors. An immediate passage to the inductive limit gives:

Corollary 2. *Let X be an arbitrary topological space. There exists a spectral functor on the category $\mathbf{C}^X$ of sheaves of abelian groups on X, converging to the graded functor $(H^n(X,F))$, whose initial term is given by*

$$E_2^{p,q}(F) = \check{H}^p(X, H^q(F))$$

($H^q(F)$ being the presheaf defined in Theorem 3.5.1). We have

$$E_2^{0,q}(F) = 0, \text{ for } q > 0$$

This last formula results from the definition of $H^0(X, H^q(F))$ and from the following result:

3.8.2 Lemma. *Let U be a neighborhood of x and let $c^q \in H^q(U,F)$. Then there exists a neighborhood $V \subseteq U$ of x such that the image of c^q in $H^q(U,F)$ vanishes.*

To see this it suffices to take an injective resolution of $F|U$, with C being the corresponding complex, and to represent c^q by an element of $H^q(\Gamma(U,C))$, defined by a cocycle $z \in \Gamma(U, C^q)$; in accordance with the acyclicity of C in dimension q, the restriction of z to an appropriate neighborhood V of x is a coboundary, whence the result. We also

[t]Translator's note: This notation has not appeared before; it seems likely that he means that $U_{i_0,i_1,\ldots,i_p} = \bigcap U_{i_j}$.

note that $C|V$ is an injective resolution of $F|V$ by virtue of Proposition 3.1.3, and thus $H^q(V,F) = H^q(\Gamma(C|V))$.

The spectral sequence of Corollary 2 gives natural transformations

$$\check{H}^p(X,F) \longrightarrow H^p(X,F)$$

and the formula $E_2^{0,1} = 0$ shows that:

Corollary 3. *The preceding natural transformations are equivalences if $p = 0$ or 1 (which we already knew) and monomorphisms if $p = 2$.*

We recall, moreover, that *if X is paracompact* then $\check{H}^p = H^p$ and, more precisely, *the canonical natural transformations above are then equivalences.* In effect, we show in this case (thanks to the fact that the sheaf associated to the presheaf $H^q(F)$ vanishes when $q > 0$) that $E_2^{p,q} = 0$ for $q > 0$; cf. [9].

Corollary 3 can be generalized as follows: *if $\check{H}^p(X, H^q(F)) = 0$ for $0 < q < n$, then the natural transformation $\check{H}^i(X,F) \longrightarrow H^i(X,F)$ is an equivalence for $i \leq n$ and a monomorphism for $i = n+1$.* From this we infer (with H. Cartan):

Corollary 4. *Let $\mathbf{U}$ be a set of open sets forming a base for the topology of X. Let F be an abelian sheaf on X such that, for every non-empty sequence $(U_1, \ldots, U_k)$ of open sets of $\mathbf{U}$, their intersection U satisfies $\check{H}^i(U,F) = \{0\}$ for $i > 0$. Then we also have $H^1(U,F) = \{0\}$, and for every open subset V of X, the natural transformation $\check{H}^i(V,F) = H^i(V,F)$ is an equivalence.*

It is sufficient to prove that $H^i(U,F) = \{0\}$ since V admits arbitrarily fine covers $\mathbf{R}$ by open sets from $\mathbf{U}$, and we conclude from Corollary 1 that for such an $\mathbf{R}$ the homomorphism $\check{H}^i(\mathbf{R},F) \longrightarrow H^i(V,F)$ is an isomorphism, which proves at the same time ($\mathbf{R}$ being arbitrarily fine) that $\check{H}^i(V,F) \longrightarrow H^i(V,F)$ is an isomorphism. To prove $H^i(U,F) = \{0\}$ we prove by induction on n that $H^i(U,F) = \{0\}$ for $0 < i \leq n$ and for every U as described. This is trivial if $n = 0$. We assume $n \geq 1$ and that the statement holds for $n' = n-1$. There are arbitrarily fine covers $\mathbf{R}$ of U by open sets from $\mathbf{U}$; for such an $\mathbf{R}$, we have $C(\mathbf{R}, H^q(F)) = 0$ for $0 < q < n$ by induction, a fortiori, $H^p(\mathbf{R}, H^q(F)) = 0$ for such q, whence $\check{H}^p(U, H^q(F)) = 0$ for such U, which, by virtue of the paragraph preceding Corollary 4, implies that $H^n(U,F) = \check{H}^n(U,F)$, which is zero.

Corollary 4 applies, for example, to the case that X is an algebraic variety equipped with its Zariski topology, $\mathbf{U}$ is the set of affine open sets in X and F is a coherent algebraic sheaf over X [15]. According to [15], the affine open sets form a basis for the topology of X, and the intersection of two affine open sets is an affine open set. If U is an affine open set, we have $\check{H}^i(U,F) = 0$ for $i > 0$. Thus we have $H^i(X,F) = \check{H}^i(X,F)$; moreover, Corollary 1 above shows that we can calculate the $H^i(X,F)$ using a *single* arbitrarily chosen cover of X by affine open sets.

Remark. There are other cases besides the one in Theorem 3.8.1 in which the Leray spectral sequence is valid. The best known is the case of a *locally finite cover of X, assumed to be paracompact, by closed sets* (the case assumed by Leray); the simplest way to treat it is as above for open covers, thanks to the fact that the restriction of a soft sheaf to a closed subset is still a soft sheaf (replacing Proposition 3.1.3). Another case, handled by Godement using a different method, is that of a *finite cover of X by closed sets* (without assuming paracompactness). When both hypotheses hold, the two spectral sequences fortunately coincide.

3.8.3 Example. To conclude this section, we will describe a simple example in which the monomorphism $\check{H}^2(X,F) \longrightarrow H^2(X,F)$ is not an isomorphism, and in which we even have $\check{H}^2(X,F) = 0$ and $H^2(X,F) \neq 0$. Since we can infer from Corollary 2 of Theorem 3.8.1 an exact sequence

$$0 \longrightarrow \check{H}^2(X,F) \longrightarrow H^2(X,F) \longrightarrow \check{H}^1(X,H^1(F)) \longrightarrow 0$$

it suffices to *show a case in which $H^2(X,F) \neq 0$ and $H^2(X,F) \longrightarrow \check{H}^1(X,H^1(F))$ is an isomorphism.*

Let X be an irreducible space (cf. end of 3.3), Y_1 and Y_2 be two irreducible closed subsets of X which meet at exactly two points x_1 and x_2 (for example, two intersecting circles in the plane equipped with the Zariski topology), and Y be their union. By abuse of notation, we denote by $\mathbf{Z}$ the constant sheaf of integers over X, and we consider, using the notation of 3.5, the exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z}_{\complement Y} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}_Y \longrightarrow 0$$

We claim that the sheaf $F = \mathbf{Z}_{\complement Y}$ satisfies the desired conditions. First, since $H^p(X,\mathbf{Z}) = 0$ for $p > 0$ according to the end of 3.3, we have $H^2(X,\mathbf{Z}_{\complement Y}) = H^1(X,\mathbf{Z}_Y) = H^1(Y,\mathbf{Z})$. We show that this group is not 0, and specifically that it is isomorphic to $\mathbf{Z}$. In fact, we have a natural monomorphism from the constant sheaf $\mathbf{Z}$ over Y to the direct sum of the sheaves $\mathbf{Z}_{Y_1}$ and $\mathbf{Z}_{Y_2}$, from which we have an exact sequence of sheaves over Y:

$$0 \longrightarrow \mathbf{Z}_Y \longrightarrow (\mathbf{Z}_{Y_1} \oplus \mathbf{Z}_{Y_2}) \longrightarrow \mathbf{Z}_{Y_1 \cap Y_2} \longrightarrow 0$$

Since Y_i is irreducible, we have $H^p(Y,\mathbf{Z}_{Y_1}) = H^p(Y_i,\mathbf{Z}) = 0$ for $p > 0$, from which we get $H^1(Y,\mathbf{Z}) = \Gamma(\mathbf{Z}_{Y_1 \cap Y_2})/\operatorname{Im}(\Gamma(\mathbf{Z}_{Y_2}))$. This is the cokernel of the homomorphism of groups $\mathbf{Z}^2 \longrightarrow \mathbf{Z}^2$ given by $(n_1,n_2) \mapsto (n_1 - n_2, n_1 - n_2)$, that is, a group isomorphic to $\mathbf{Z}$ from which we get

$$H^2(X,\mathbf{Z}_{\complement Y}) = H^1(X,\mathbf{Z}) = \mathbf{Z}$$

It remains to be proved that $\check{H}^1(X,\check{H}^1(\mathbf{Z}_{\complement Y}))$ is isomorphic to $\mathbf{Z}$ (since, given the preceding relation, the epimorphism $\check{H}^2(X,F) \longrightarrow H^1(X,H^1(F))$ will necessarily be an isomorphism).

We calculate $H^1(\mathbf{Z}_{\complement Y})$; for any open set V, $H^1(V, Z_{\complement Y})$ can be calculated using the following exact sequence of sheaves *over* V:

$$0 \longrightarrow \mathbf{Z}_{\complement Y} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}_{Y'} \longrightarrow 0$$

in which we have set $Y' = Y \cap V$. Since V is also irreducible, we see $H^1(V, \mathbf{Z}_{\complement Y}) = H^0(Y', \mathbf{Z})/\operatorname{Im} H^0(V, \mathbf{Z}) = \overline{H}^0(Y', \mathbf{Z})$ where the last group denotes the *reduced* integer cohomology group in dimension 0, which here is the free abelian generated by the connected components of Y' modulo the diagonal subgroup. Here $Y' = Y \cap V$ is an open subspace of $Y = Y_1 \cup Y_2$, and thus has 0, 1, or 2 connected components, the last case arising exactly in the case that V meets both Y_1 and Y_2 without meeting their intersection; thus we have $H^1(V, \mathbf{Z}_{\complement Y}) = 0$, except in that last case.

To calculate $H^1(X, H^1(F)) = \varinjlim H^1(\mathbf{U}, H^1(F))$, we can restrict ourselves to the covers $\mathbf{U} = (U_x)_{x \in X}$ such that each U_x meets at most one of the two closed sets Y_1 and Y_2, except if x is one of the two points x_1 or x_1 of $Y_1 \cap Y_2$, in which case we assume it does not contain the other. For such a $\mathbf{U}$ we immediately see that $C^0(\mathbf{U}, H^1(F)) = 0$, therefore $H^1(\mathbf{U}, H^1(F))$ is identified with the group $Z^1(\mathbf{U}, H^1(F))$ of the 1-cocycles of $\mathbf{U}$ with coefficients in $H^1(F)$, or $(f_{x,y})_{x,y \in X}$. But we have $\Gamma(U_x \cap U_y, H^1(F)) = 0$ unless $x = x_1$ and $y = x_2$ or vice versa. From this we have $C^1(\mathbf{U}, H^1(F)) \cong \mathbf{Z}^2$, and we readily see that the cocycles are identified with the pairs $(n, -n)$, where $(n = f_{x_1,x_2})$. We thus have $H^1(\mathbf{U}, H^1(F)) = \mathbf{Z}$ whence we see immediately that at the limit $\check{H}^1(H^1(F)) = \mathbf{Z}$, which completes the proof.

3.9 Acyclicity criteria by the method of covers

Let X be a topological space and $\mathcal{S}$ be a non-empty set of subsets of X. For every $A \in \mathcal{S}$ we assume given a non-empty set $\mathcal{R}(A)$ of covers $\mathbf{R}(A)$ by sets from $\mathcal{S}$ and their finite intersections. We assume that if $B \in \mathbf{R} \in \mathcal{R}(A)$, then the trace $\mathbf{R}_B$ of $\mathbf{R}$ on B belongs to $\mathcal{R}(B)$. We assume, in addition, that we have one of the three following conditions (which allow us to write the Leray-Cartan spectral sequence for each of the covers $\mathbf{R} \in \mathcal{R}(A)$ with $A \in \mathcal{S}$): (1) the $A \in \mathcal{S}$ are open; (2) the $A \in \mathcal{S}$ are closed and the covers $\mathbf{R} \in \mathcal{R}(A)$ are finite; (3) the $A \in \mathcal{S}$ are closed, X is paracompact, and the $\mathbf{R} \in \mathcal{R}(A)$ are locally finite.

3.9.1 THEOREM. *Under the preceding conditions, we assume given an abelian sheaf F over X and a natural number $n \geq 0$. We assume that the following conditions are satisfied:*

$A(n)$: $H^i(\mathbf{R}, F) = 0$ for $1 \leq i \leq n$ and every $\mathbf{R} \in \mathcal{R}(A)$, $A \in \mathcal{S}$.

$B(n-1)$: For every $A \in \mathcal{S}$, every $c^i \in H^i(A, F)$ (with $1 \leq i \leq n-1$) there exists a finite subset L of $\mathcal{R}(A)$ such that if $\mathbf{R}^L$ denote the "intersection" cover of the covers $\mathbf{R} \in L$, the restriction of c^i to any set $B \in \mathbf{R}^L$ vanishes.

Under these conditions, for every $A \in \mathcal{S}$, we have $H^i(A, F) = \{0\}$ for $1 \leq i \leq n-1$, and if $c^n \in H^n(A, F)$, then $c^n = 0$ if and only if we can find a finite subset L of $\mathcal{R}(A)$ such that the restriction of c^n to any $B \in \mathbf{R}^L$ is zero.

We state right away the most interesting corollaries:

COROLLARY 1. *Using the notation of Theorem 3.9.1, in order for $H^i(A,F) = \{0\}$ for every $A \in \mathcal{S}$ and $1 \leq i \leq n$, it is necessary and sufficient for conditions $A(n)$ and $B(n)$ to be satisfied.*

The sufficiency results immediately from the theorem. Conversely, we assume $H^i(A,F) = \{0\}$ for $A \in \mathcal{S}$ and $1 \leq i \leq n$. Then $B(n)$ is trivially proved. We prove $A(n)$ as follows. The Leray spectral sequence for the cover $\mathbf{R}(A)$ (Theorem 3.8.1 Remark 3.8.2) converges to $H^*(A,F)$ and has as its initial term $E_2^{p,q} = H^p(\mathbf{R}, H^q(F))$, which is zero if $1 \leq q \leq n$, since $C(\mathbf{R}, H^q(F)) = 0$ for those values of q (since the finite intersections of sets in $\mathbf{R}$ belong to $\mathcal{S}$). Classically we can conclude that $H^i(A,F) = H^i(\mathbf{R},F)$ for $0 \leq i \leq n$, whence $H^i(\mathbf{R},F) = 0$ for such i.

COROLLARY 2. *Assume (using the notation of 3.9.1) that for every $A \in \mathcal{S}$ and every open cover of A, we can find a finer cover of the form $\mathbf{R}^L$, where L is a finite subset of $\mathcal{R}(A)$. Then for $H^i(A,F) = \{0\}$ for every $A \in \mathcal{S}$ and $1 \leq i \leq n$, it is necessary and sufficient that $H^i(\mathbf{R},F) = 0$ for every $\mathbf{R} \in \mathcal{R}(A)$ where $A \in \mathcal{S}$ and $1 \leq i \leq n$.*

Condition $B(n)$ is proved (for any $n > 0$) using Lemma 3.8.2, thus it suffices to apply Corollary 1.

COROLLARY 3. *Suppose the preliminary condition of the preceding corollary is satisfied. Assume in addition that the nerves of the covers $\mathbf{R} \in \mathcal{R}(A)$ have dimension at most n. Then the equivalent conditions of the preceding corollary imply $H^i(A,F) = 0$ for $A \in \mathcal{S}$ and every $i > 0$.*

We automatically have $H^i(\mathbf{R},F) = 0$ for $\mathbf{R} \in \mathcal{R}(A)$ and $i > n$, thus for every i, i.e. condition $A(m)$ will be proved for *(*every) m, so it is sufficient to apply the corollary with sufficiently large m.

PROOF. of Theorem 3.9.1. We will proceed by induction on n. Since the result is trivial when $n = 0$, we assume that $n \geq 1$ and we also assume the theorem holds for integers less than n. From the induction hypothesis, $H^i(A,F) = \{0\}$ for $1 \leq i \leq n-1$, so it remains to prove the vanishing of $c^n \in H^n(A,F)$ under the condition that there is a finite subset L of $\mathcal{R}(A)$ such that the restriction of c^n to every $B \in \mathbf{R}^L$ is zero. Let k be the number of elements of L, we proceed by induction on k. The conclusion is trivial if $k = 0$; we prove it for $k = 1$. By assumption, there is an $\mathbf{R} \in \mathcal{R}(A)$ such that the restriction of c^n to every $B \in \mathbf{R}$ vanishes. Since the $B \in \mathbf{R}$ are in $\mathcal{S}$, we have $H^i(B,F) = \{0\}$ for $1 \leq i \leq n-1$, then the term $E_2^{p,q}$ of the Leray spectral sequence relative to $\mathbf{R}$ vanishes for $1 \leq q \leq n-1$. We classically derive an exact sequence $H^n(\mathbf{R},F) \longrightarrow H^n(A,F) \longrightarrow H^0(\mathbf{R}, H^n(F)) \longrightarrow \cdots$. Since the first time vanishes as a result of $A(n)$, the second homomorphism is injective, therefore by assumption the image of c^n under the latter homomorphism vanishes and hence $c^n = 0$.

We now assume $k \geq 2$ and the that the conclusion holds for integers less than k. Let $L = (\mathbf{R}^1, \ldots, \mathbf{R}^k)$. It suffices to prove, as we have seen, that the restriction c^n_B of c^n to every $B \in \mathbf{R}^1$ vanishes. Now for $i = 2, \ldots, k$, the restriction $\mathbf{R}^i_B$ of $\mathbf{R}^i$ to B belongs to $\mathcal{R}(B)$; moreover, the restriction of c^n_B to any set belonging to the intersection of the covers $\mathbf{R}^i_B$ $(2 \leq i \leq k)$ vanishes by assumption on c^n. Applying our induction hypothesis, for $k-1$, to c^n_B and to B, we find that $c^n_B = 0$, which completes the proof of the theorem.

3.9.2 Proposition. *The preliminary assumption of Corollary 2 is satisfied in the following case: X is quasi-compact, the $A \in \mathcal{S}$ are closed, $X \in \mathcal{S}$, the $\mathbf{R} \in \mathcal{R}(X)$ are finite, and for two distinct points $x, y \in X$, there exists a $\mathbf{R} \in \mathcal{R}(X)$ no element of which contains both x and y.*

(We will say that a space is *quasi-compact* if it satisfies the axiom of open covers of compact spaces, without necessarily being separated.[u])

Proof. For a cover $\mathbf{R}$ of X and $x \in X$ we denote by $E_x(\mathbf{R})$ (*star* of $\mathbf{R}$ on x) the union of the $A \in \mathbf{R}$ that contain x. Let $O_x(\mathbf{R})$ be the complement of the union of the $A \in \mathbf{R}$ that do not contain x. We thus have $x \in O_x(\mathbf{R}) \subseteq E_x(\mathbf{R})$; moreover, for every $y \in O_x(\mathbf{R})$ we have $E_y(\mathbf{R}) \subseteq E_x(\mathbf{R})$. If $\mathbf{R}$ is a finite cover of X by closed sets, $O_x(\mathbf{R})$ is an *open neighborhood* of x. Under the conditions of the proposition, let $\mathbf{U}$ be an open cover of X, let $x \in X$, and $U_x \in \mathbf{U}$ with $x \in U_x$. The intersection of the $E_x(\mathbf{R})$ for $\mathbf{R} \in \mathcal{R}(X)$ is, by assumption, reduced to x, and from this we conclude, by quasi-compactness, that there is a finite subset L_x of $\mathcal{R}(X)$ such that the intersection of the E_x of $\mathbf{R}$, for $\mathbf{R} \in L_x$, i.e. $E_x(\mathbf{R}^{L_x})$, is contained in U_x. The $O_x(\mathbf{R}^{L_x})$ form an open cover of X, therefore there is a finite set $Y \subseteq X$ such that the $O_x(\mathbf{R}^{L_x})$ corresponding to the $x \in Y$ cover X. Let L be the union of the L_x for $x \in Y$. We claim that the cover $\mathbf{R}^L$ of X is finer than $\mathbf{U}$. Let $A \in \mathbf{R}^L$ be non-empty and $a \in A$. There exists $x \in Y$ such that $a \in O_x(\mathbf{R}^{L_x})$, whence $E_a(\mathbf{R}^{L_x}) \subseteq E_x(\mathbf{R}^{L_x}) \subseteq U_x$, and, a fortiori, $A \subseteq E_a(\mathbf{R}^L) \subseteq E_a(\mathbf{R}^{L_x}) \subseteq U_x$, *a fortiori* $A \subseteq E_a(\mathbf{R}^L) \subseteq E_a(\mathbf{R}^{L_x}) \subseteq U_x$ which establises our claim. Applying this result, for every $A \in \mathcal{S}$ to the set of covers of A, induced by the covers $\mathbf{R} \in \mathcal{R}(X)$, the desired conclusion follows.

The most striking application of Corollary 3 is the one in which X is the compact cube $0 \leq x_i \leq 1$ of $\mathbf{R}^m$ with $\mathcal{S}$ the family of compact cubes A of the type $a_i \leq x_i \leq b_i \subseteq X$, $\mathcal{R}(A)$ being the family of covers of A defined by hyperplanes parallel to the coordinate hyperplanes. To show that $H^i(A, F) = 0$ for $i > 0$ and every A, it suffices to show that for every A and every section $f \in F$ over $A_1 \cap A_2$, we have $f = f_1 - f_2$, where f_i is a section of F on A_i. This is the reduction performed by H. Cartan in his proof of the fundamental theorems on Stein varieties [5].

Remark. If $n = 1$, Theorem 3.9.1 is still meaningful and can easily be proved directly if we assume that F is a sheaf of not necessarily abelian groups. This makes it possible to

[u] Translator's note: "Quasicompact" is what everybody who has not been polluted by Bourbaki calls "compact".

simplify the proof of the theorem [5, Chapter XVII] on invertible holomorphic matrices.

3.10 Passage to the limit in sheaf cohomology

We will give only two results of this type (one of which we will use in Chapter 5, Section 7), special cases of the following general result in homological algebra:

3.10.1 PROPOSITION. *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be abelian categories. We assume that every object of* $\mathbf{C}$ *is isomorphic to a subobject of an injective, and that* $\mathbf{C}'$ *satisfies Axiom* AB 5) *(cf. 1.5), which in particular makes it possible to take inductive limits in* $\mathbf{C}'$ *(cf. Proposition 1.8). Let* $(F_i)_{i\in I}$ *be an inductive system of covariant additive functors from* $\mathbf{C}$ *to* $\mathbf{C}'$*. Let* $F = \varinjlim F_i$ *be the inductive limit functor of the* F_i*, defined by* $F(A) = \varinjlim F_i(A)$ *for every* $A \in \mathbf{C}$*. The homomorphisms* $F_i \longrightarrow F$ *define natural transformations of* ∂*-functors* $(\mathbf{R}^p F_i) \longrightarrow \mathbf{R}^p F$ *from which we derive a natural transformation of* ∂*-functors*

$$\varinjlim \mathbf{R}^p F_i(A) \longrightarrow \mathbf{R}^p F(A) \tag{3.10.1}$$

(the coboundary homomorphisms for the sequence of functors $\varinjlim \mathbf{R}^p F_i$ *are defined as the inductive limit of the coboundary homomorphisms relative to the* $\mathbf{R}^p F_i$*). The natural transformations (3.10.1) are equivalences.*

To see this, it suffices to take an injective resolution $C = C(A)$ of A. Then the left hand side of (3.10.1) is $\varinjlim H^p(F_i C(A))$ and the right side is $H^p(\varinjlim F_i C(A))$. They are thus isomorphic since the functor $\varinjlim$ on the category of inductive systems on I with values in $\mathbf{C}'$ is exact (Proposition 1.8) and, in particular, commutes with taking homology of complexes.

COROLLARY 1. *Let* X *be a space equipped with a paracompactifying family* Φ*. Then we have, for every abelian sheaf* F *on* X*,*

$$H^p_\Phi(X, F) = \varinjlim_U H^p(X, F_U)$$

the inductive limit taken over the set of open sets $U \in \Phi$ *directed by containment. (*F_U *denotes the sheaf over* X *whose restriction to* U *is* $F|U$ *and to* $\complement U$ *is zero.)*

By Theorem 3.5.1 we have $H^p(X, F_U) = H^p_{\Phi_U}(U, F)$, where Φ_U is the set of closed subsets of U. Setting $\Gamma_{\Phi_U}(F) = H^0(X, F_U)$, we can also write $H^p(X, F_U) = R^p\Gamma_{\Phi_U}(F)$ (using Proposition 3.1.3), from which we have by Proposition 3.10.1: $\varinjlim H^p(X, F_U) = R^p\Gamma_\Phi(F)$, since $\varinjlim \Gamma_{\Phi_U}(F) = \Gamma_\Phi(F)$. Q.E.D.

The preceding corollary can be useful for reducing the cohomology with "support in Φ" to cohomology with arbitrary support and was pointed out to me by H. Cartan.

Corollary 2. *Let X be a topological space and Y be a subspace of X admitting a basis of paracompact neighborhoods (it suffices, for example, that X be metrisable or locally compact paracompact). Then for every abelian sheaf F over X, we have*

$$H^p(Y,F) = \varinjlim H^p(U,F)$$

the limit taken over the decreasing directed set of open neighborhoods $U \subseteq Y$.

In fact, this follows from the assumption that $H^0(Y,F) = \varinjlim H^0(U,F)$ [11, 2.2.1]. The derived functors of $F \mapsto H^0(U,F)$ are the $H^p(U,F)$ so that Corollary 2 is a special case of the proposition. We should note that we also have $H^0(Y,F) = \varinjlim H^0(U,F)$ and therefore Corollary 2 follows if Y is *closed* and is contained in a *single* paracompact neighborhood (a proof analogous to the one in [11]); in *loc. cit.* we also find a simple counter-example (with $p = 0$) for the case in which no hypothesis of paracompactness is made.

By way of completeness, we indicate the following result without proof, a special case of general results on projective systems. Let X be a locally compact space. We consider the increasing directed set of the relatively compact open subspaces U of X[v]. Then for every abelian sheaf F over X, the restriction homomorphism $H^p(X,F) \longrightarrow H^p(U,F)$ define canonical homomorphisms (which are obviously natural transformations of ∂-functors):

$$H^p(X,F) \longrightarrow \varprojlim H^p(U,F) \tag{3.10.2}$$

which are obviously bijective for $p = 0$.

3.10.2 Proposition. *Suppose that the locally compact space X is σ-compact* [w]. *Then the homomorphisms 3.10.2 (where the projective limit is taken over the increasing directed set of relatively compact open subspaces $U \in X$) are surjective. If $p > 1$, in order that it to be bijective, it is sufficient that for every relatively compact open subspace U, there exist another $V \supseteq U$ such that for every relatively compact open subset $W \supseteq V$, the image in $H^{p-1}(U,F)$ of $H^{p-1}(W,F)$ by the restriction homomorphism is identical to that of $H^(p-1)(V,F)$. If $p = 1$, for the homomorphism $H^1(X,F) \longrightarrow \varprojlim H^1(U,F)$ to be bijective, it suffices that to equip $H^0(U,F)$ with topologies of complete metrizable topological groups in such a way that the restriction homomorphisms are continuous, and that for every relatively compact open subset U there exists $V \supseteq U$ such that for every relatively compact subset $W \supseteq V$, the image of $H^0(W,F)$ in $H^0(U,F)$ is dense in the image of $H^0(V,F)$.*

[v] Translator's note: This means that their closures are compact

[w] Translator's note: The original phrase, "d'énombrable à l'infini" was unknown to us. We thank Jonathan Chiche for pointing out the reference to the paper *Cohomologies à coefficients dans un faisceau* by Michel Zisman, available at `http://archive.numdam.org/ARCHIVE/SD/SD_1957-1958__11_1/SD_1957-1958__11_1_A8_0/SD_1957-1958__11_1_A8_0.pdf`. Theorem 4.1 of that paper says that a locally compact space that is 'dénombrable à l'infini' [i.e. σ-compact] is paracompact."

(Of course, in this statement, we could replace relatively compact open subsets by compact subsets.) This proposition, which can be proved by a process of approximation à la Mittag-Loeffler, is essential, for example, in the proof of the fundamental theorems on Stein varieties [5]. For $p = 1$ it remains true if F is a sheaf of not-necessarily abelian groups taking the form: if the stated approximation condition (in the case $p = 1$) holds, every element of the left hand side of 3.10.2 whose image is the identity element is the identity.[x]

[x]Translator's note: This seems to be an especially obscure way of saying that the homomorphism of 3.10.2 is injective.

Chapter 4

Ext of sheaves of modules

4.1 The functors $\mathrm{Hom}_{\mathbf{O}}(A, B)$ and $\mathbf{Hom}_{\mathbf{O}}(A, B)$

Let X be a topological space equipped with a sheaf $\mathbf{O}$ of unital rings, and let $\mathbf{C}^{\mathbf{O}}$ be the abelian category of left $\mathbf{O}$-modules over X (cf. 3.1). If A and B are two $\mathbf{O}$-modules, we denote by $\mathrm{Hom}_{\mathbf{O}}(A, B)$ the group of $\mathbf{O}$-homomorphisms of A to B (this group is, in fact, a module over the center of $\Gamma(\mathbf{O})$). If U is an arbitrary subset of X, we set

$$\mathrm{Hom}_{\mathbf{O}}(U; A, B) = \mathrm{Hom}_{\mathbf{O}|U}(A|U, B|U)$$

(where $F|U$ denotes, as usual, the restriction of the sheaf F to U). Of course, if $V \subseteq U$, we have a natural homomorphism $\mathrm{Hom}_{\mathbf{O}}(U; A, B) \longrightarrow \mathrm{Hom}_{\mathbf{O}}(V; A, B)$, and we can readily see that if we restrict ourselves to open subsets $U \subseteq X$, the groups $\mathrm{Hom}_{\mathbf{O}}(U; A, B)$ form a *sheaf* on X denoted $\mathbf{Hom}_{\mathbf{O}}(A, B)$. Thus we have, by definition

$$\Gamma(U, \mathbf{Hom}_{\mathbf{O}}(A, B)) = \mathrm{Hom}_{\mathbf{O}}(U; A, B) \tag{4.1.1}$$

and specifically, if we set $U = X$:

$$\mathrm{Hom}_{\mathbf{O}}(A, B) = \Gamma(\mathbf{Hom}_{\mathbf{O}}(A, B)) \tag{4.1.2}$$

Moreover, $\mathbf{Hom}_{\mathbf{O}}(A, B)$ can also be considered a sheaf of modules over the *center* $\mathbf{O}^\natural$ of $\mathbf{O}$.

Recall that $\mathrm{Hom}_{\mathbf{O}}(A, B)$ is a *left exact additive functor* in both arguments $A, B \in \mathbf{C}^{\mathbf{O}}$ with values in abelian groups. We conclude from this that $\mathbf{Hom}_{\mathbf{O}}(A, B)$ can also be regarded as a left exact functor in the two arguments $A, B \in \mathbf{C}^{\mathbf{O}}$ with values in the category $\mathbf{C}^X$ of abelian sheaves over X (or even with values in the category $\mathbf{C}^{\mathbf{O}^\natural}$). As usual, all the homomorphisms we will write will be "natural". This section gives some auxiliary properties of the desired functors as a preliminary to the study of their derived functors.

Let $A, B \in \mathbf{C}^{\mathbf{O}}$ and $x \in X$. For every open subset U containing x, there is an obvious homomorphism $\mathrm{Hom}_{\mathbf{O}}(U; A, B) \longrightarrow \mathrm{Hom}_{\mathbf{O}(x)}(A(x), B(x))$ from which, by passing to the inductive limit over the open neighborhoods of x, we get a natural homomorphism

$$\mathbf{Hom}_{\mathbf{O}}(A,B)(x) \longrightarrow \mathrm{Hom}_{\mathbf{O}(x)}(A(x),B(x)) \tag{4.1.3}$$

which we are going to study. We say that A *is of finite type at* x if we can find an open neighborhood U of x such that $A|U$ is isomorphic to a quotient of $(\mathbf{O}|U)^n$ for some finite integer $n > 0$. A is said to be *pseudo-coherent at* x if we can find an open neighborhood U of x, and *over* U an exact sequence of homomorphisms $\mathbf{O}^m \longrightarrow \mathbf{O}^n \longrightarrow A \longrightarrow 0$ (where m and n are finite integers > 0).[y] We say that A is of finite type, respectively pseudo-coherent, if it is so at every point. Thus:

4.1.1 Proposition. *The homomorphism (4.1.3) is a monomorphism if A is of finite type at x and an isomorphism if A is pseudo-coherent at x.*

Suppose that A is of finite type at x. Then by restricting ourselves as necessary to a suitable open neighborhood of x, we have an exact sequence $\mathbf{O}^n \longrightarrow A \longrightarrow 0$, from which we get an exact sequence $0 \longrightarrow \mathbf{Hom}_{\mathbf{O}}(A,B) \longrightarrow \mathbf{Hom}_{\mathbf{O}}(\mathbf{O}^n, B)$ and an exact sequence $\mathbf{O}^n(x) \longrightarrow A(x) \longrightarrow 0$, and whence an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{O}(x)}(A(x),B(x)) \longrightarrow \mathrm{Hom}_{\mathbf{O}(x)}(\mathbf{O}^n(x),B(x))$$

From this we deduce a homomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Hom}_{\mathbf{O}}(A,B)(x) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathbf{O}(x)}(\mathbf{O}^n,B)(x) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathbf{O}(x)}(A(x),B(x)) \arrow[r] & \mathrm{Hom}_{\mathbf{O}(x)}(\mathbf{O}^n(x),B(x))
\end{tikzcd}$$

Now we have $\mathbf{Hom}_{\mathbf{O}}(\mathbf{O}^n, B) = \mathbf{Hom}_{\mathbf{O}}(\mathbf{O}, B)^n = B^n$ since there is a canonical isomorphism

$$\mathbf{Hom}_{\mathbf{O}}(\mathbf{O},B) \cong B \tag{4.3.4}$$

The groups in the last column of the diagram are thus identified, respectively, with $B(x)^n$ and $\mathrm{Hom}_{\mathbf{O}(x)}(\mathbf{O}(x), B(x))^n = B(x)^n$, so the last vertical homomorphism is an isomorphism, from which we readily conclude that the first homomorphism is a monomorphism. Now suppose that A is pseudo-coherent at x, that is (by restricting ourselves if necessary to a

[y] Translator's note: These restrictions to *positive* integers serve no purpose whatever, although they also cause no harm.

suitable neighborhood of x), we have an exact sequence $\mathbf{O}^m \longrightarrow \mathbf{O}^n \longrightarrow A \longrightarrow 0$. Using the left exactness of the Hom functors, we derive a homomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Hom}_{\mathbf{O}}(A,B)(x) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathbf{O}}(\mathbf{O}^n,B)(x) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathbf{O}}(\mathbf{O}^m,B)(x) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathbf{O}(x)}(A(x),B(x)) \arrow[r] & \mathrm{Hom}_{\mathbf{O}(x)}(\mathbf{O}^n(x),B(x)) \arrow[r] & \mathrm{Hom}_{\mathbf{O}(x)}(\mathbf{O}^m(x),B(x))
\end{tikzcd}$$

We have already noted that the last two vertical homomorphisms are bijective, from which it follows immediately that the first one is an isomorphism.

Now suppose that B is injective in the abelian category $\mathbf{C}^{\mathbf{O}}$. Can we conclude from this that $B(x)$ is an injective $\mathbf{O}(x)$-module? By Lemma 1 of 1.10, it suffices to show that for any left ideal $\mathbf{a}$ of $\mathbf{O}(x)$, any $\mathbf{O}(x)$ homomorphism from $\mathbf{a}$ to $B(x)$ extends to an $\mathbf{O}(x)$ homomorphism from $\mathbf{O}(x)$ to $B(x)$. Since the restriction of an injective $\mathbf{O}$-module to an open set is still injective (Proposition 3.1.3), it is sufficient, given a homomorphism $\mathbf{a} \longrightarrow B(X)$, to find an open neighborhood U of x, a submodule A of $\mathbf{O}|U$, and a homomorphism u from A to $B|U$ such that $A(x) = \mathbf{a}$ and such that the homomorphism from $A(x)$ to $B(x)$ defined by u is the given homomorphism. Moreover, if A is pseudo-coherent at x, the existence of u follows automatically from Proposition 4.1.1. We then prove:

LEMMA 1. *Suppose that* $\mathbf{O}$ *is a coherent sheaf of rings [15, Chapter I.2]. Let M be a* $\mathbf{O}$-*coherent module ("coherent sheaf of modules" according to the terminology of [15]) and let* $\mathbf{n}$ *be a submodule of finite type of $M(x)$. Then there exists a neighborhood U of x and a coherent subsheaf N of M on U such that $N(x) = \mathbf{n}$.*

Let (n_i) be a finite system of generators of $\mathbf{n}$ $(1 \leq i \leq k)$. For every i, we have $n_i \in M(x)$ so that $n_i = f_i(x)$ where f_i is a section of M defined in an open neighborhood of x. We can assume that all the f_i are defined on the same open neighborhood U. Then they define a homomorphism $\mathbf{O}^k \longrightarrow M$ over U. We will take for N the image of this homomorphism, which is coherent, being a submodule of finite type of the coherent sheaf M. Taking into account the comments preceding the statement of this lemma, we find:

4.1.2 PROPOSITION. *Suppose that* $\mathbf{O}$ *is a coherent sheaf of left Noetherian rings. Then for every injective* $\mathbf{O}$-*module B and every $x \in X$, $B(x)$ is an injective* $\mathbf{O}(x)$-*module.*

Finally, we point out:

4.1.3 PROPOSITION. *Let A and B be two* $\mathbf{O}$-*modules. If B is injective, then* $\mathbf{Hom}_{\mathbf{O}}(A,B)$ *is a flabby sheaf (cf. 3.3).*

In effect, a section of this sheaf over the open set U is identified with a homomorphism from the $\mathbf{O}$-module A_U to B (the notation A_U is as in 3.5), therefore (since B is injective) extends to a homomorphism from A to B, i.e., to a section of $\mathbf{Hom}_{\mathbf{O}}(A,B)$ over every X.

4.2 The functors $\mathrm{Ext}^p_{\mathbf{O}}(X;A,B)$ and $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$ and the fundamental spectral sequence.

By virtue of the corollary to Proposition 3.1.1, we can take the derived functors of any additive covariant functor defined on $\mathbf{C}^{\mathbf{O}}$. In particular, this makes it possible to construct in the usual way the functors $\mathrm{Ext}^p(A,B)$ in $\mathbf{C}^{\mathbf{O}}$ as right derived functors of the functor $B \mapsto \mathrm{Hom}(A,B)$ in $\mathbf{C}^{\mathbf{O}}$ with values in the category of abelian groups. To avoid notational confusion, we are going, however, to denote these functors $\mathrm{Ext}^p_{\mathbf{O}}(X;A,B)$ (indicating in the notation both the space X and the sheaf $\mathbf{O}$ of rings). We set, for every subset $U \subseteq X$:

$$\mathrm{Ext}^p_{\mathbf{O}}(U;A,B) = \mathrm{Ext}^p_{\mathbf{O}|U}(U;A|U,B|U) \tag{4.2.1}$$

Given Proposition 3.1.3 and the fact that the functor $B \mapsto B|U$ is exact, we see that the $\mathrm{Ext}^p_{\mathbf{O}}(U;A,B)$ are the right derived functors of the functor $B \mapsto \mathrm{Hom}_{\mathbf{O}}(U;A,B)$ on $\mathbf{C}^{\mathbf{O}}$. The natural transformations $\mathrm{Hom}_{\mathbf{O}}(U;A,B) \longrightarrow \mathrm{Hom}_{\mathbf{O}}(V;A,B)$ therefore define transformations for the right derived functors

$$\mathrm{Ext}^p_{\mathbf{O}}(U;A,B) \longrightarrow \mathrm{Ext}^p_{\mathbf{O}}(V;A,B), \qquad V \subseteq U \tag{4.2.2}$$

thanks to which the system of the $\mathrm{Ext}^p_{\mathbf{O}}(U;A,B)$, varying over open U, becomes an *abelian presheaf* on X, denoted $\mathrm{Ext}^p_{\mathbf{O}}(-;A,B)$.

We can also consider the right derived functors of $\mathbf{Hom}_{\mathbf{O}}(A,B)$ with respect to B; we denote them by $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$. Here, given Proposition 3.1.3 and the obvious relation $\mathbf{Hom}_{\mathbf{O}}(A,B)|U = \mathbf{Hom}_{\mathbf{O}|U}(A|U,B|U)$, we find isomorphisms

$$\mathbf{Ext}^p_{\mathbf{O}}(A,B)|U \cong \mathbf{Ext}^p_{\mathbf{O}|U}(A|U,B|U) \tag{4.2.3}$$

(for this reason, we do not bother to specify the ambient space in the notation). Loosely, *the functors* $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$ *are local* (in contrast to the functors $\mathrm{Ext}^p_{\mathbf{O}}(X;A,B)$, which are essentially "global"). By Lemma 3.7.2, $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$ can also be considered as the sheaf associated to the presheaf $\mathrm{Ext}^p_{\mathbf{O}}(-;A,B)$ defined above.

The considerations at the end of 2.3 show that the $\mathrm{Ext}^p_{\mathbf{O}}(X;A.B)$, respectively $\mathrm{Ext}^p_{\mathbf{O}}(A,B)$, form not only *covariant cohomological functors with respect to* B but also *contravariant cohomological functors with respect to* A, thanks to the fact that for injective B, the functors $A \mapsto \mathrm{Hom}_{\mathbf{O}}(A,B)$ and $A \mapsto \mathbf{Hom}_{\mathbf{O}}(A,B)$ are exact. (The second fact can be readily verified using the first, thanks to Proposition 3.1.3.)

To complete the circle, we note that the $\mathrm{Ext}^p_{\mathbf{O}}(X;A,B)$ are modules over the center of $\Gamma(\mathbf{O})$ and that the $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$ are modules over the sheaf of rings $\mathbf{O}^\natural$, the center of $\mathbf{O}$.

The formula 4.1.2 can also be written, denoting by h_A, respectively $\mathbf{h}_A$ the covariant functor from $\mathbf{C}^{\mathbf{O}}$ to the category $\mathbf{C}$ of abelian groups, respectively $\mathbf{C}^X$, defined by

$$h_A(B) = \mathrm{Hom}_{\mathbf{O}}(A,B), \qquad \mathbf{h}_A(B) = \mathbf{Hom}_{\mathbf{O}}(A,B), \qquad h_A = \Gamma\mathbf{h}_A \tag{4.2.4}$$

Thus h_A appears as a composite functor, which is justified by Theorem 2.4.1 by virtue of Proposition 4.1.3 and the corollary to Proposition 3.3.1. We thus obtain:

4.2.1 THEOREM. *Let X be a topological space equipped with a sheaf $\mathbf{O}$ of unital rings, and let $\mathbf{C}^{\mathbf{O}}$ be the category of $\mathbf{O}$-modules over X. Let $A \in \mathbf{C}^{\mathbf{O}}$ be a fixed $\mathbf{O}$-module. Then on $\mathbf{C}^{\mathbf{O}}$ there is a cohomological spectral functor converging to the graded functor $(\mathrm{Ext}^p_{\mathbf{O}}(X; A, B))$, whose initial term is*

$$\mathrm{I}_2^{p,q}(A,B) = H^p(X, \mathbf{Ext}^q_{\mathbf{O}}(A,B)) \tag{4.2.5}$$

In particular, we derive from this *"edge homomorphisms"*

$$\begin{aligned} &H^p(X, \mathbf{Hom}_{\mathbf{O}}(A,B)) \longrightarrow \mathrm{Ext}^p_{\mathbf{O}}(X; A, B) \\ &\mathrm{Ext}^p_{\mathbf{O}}(X; A, B) \longrightarrow H^0(X, \mathbf{Ext}^p_{\mathbf{O}}(A,B)) \end{aligned} \tag{4.2.6}$$

(the latter resulting from interpreting $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$ as the sheaf associated to the presheaf $\mathrm{Ext}^p_{\mathbf{O}}(-; A, B)$) and we derive an exact 5-term sequence

$$\begin{gathered} 0 \longrightarrow H^1(X, \mathbf{Hom}_{\mathbf{O}}(A,B)) \longrightarrow \mathrm{Ext}^1_{\mathbf{O}}(X; A, B) \\ \longrightarrow H^0(X, \mathbf{Ext}^1_{\mathbf{O}}(A,B)) \longrightarrow H^2(X, \mathbf{Hom}_{\mathbf{O}}(A,B)) \longrightarrow \mathrm{Ext}^2_{\mathbf{O}}(X; A, B) \end{gathered} \tag{4.2.7}$$

This exact sequence *specifically characterizes the structure of the group* $\mathrm{Ext}^1_{\mathbf{O}}(X; A, B)$ *of the classes of* $\mathbf{O}$*-modules that are extensions of* A *(quotient modules) by* B *(submodule).*

To be able to use this spectral sequence, it is also necessary to specify the computation of the $\mathbf{Ext}^p_{\mathbf{O}}(A,B)$. Since the functor $F \mapsto F(x)$, which associates to an abelian sheaf F its point group $F(x)$ is *exact*, the $\mathbf{Ext}^p_{\mathbf{O}}(A,B)(x)$ can be considered, for fixed A and variable p, B to be forming a universal covariant cohomological functor in B, reducing to $\mathbf{Hom}_{\mathbf{O}}(A,B)(x)$ in dimension 0. Similarly, the $\mathrm{Ext}^p_{\mathbf{O}(x)}(A(x), B(x))$ form a cohomological functor in B, reducing to $\mathrm{Hom}_{\mathbf{O}(x)}(A(x), B(x))$ in dimension 0. We therefore conclude from the natural transformation (4.1.3), using the definition of universal cohomological functors, homomorphisms

$$\mathrm{Ext}^p_{\mathbf{O}}(A,B)(x) \longrightarrow \mathrm{Ext}^p_{\mathbf{O}(x)}(A(x), B(x)) \tag{4.2.8}$$

(which are characterized by the definition of homological functors that reduces to the homomorphism (4.1.3) in dimension 0). Since A is fixed, in order that these homomorphisms be isomorphisms for arbitrary p and B, it is necessary and sufficient that (i) this be true for $p = 0$ and (ii) $\mathrm{Ext}^p_{\mathbf{O}(x)}(A(x), B(x)) = 0$ for all $p > 0$ and all injective $\mathbf{O}$-modules B. Propositions 4.1.1 and 4.1.2 were clearly created expressly to verify conditions (i) and (ii). We thus get:

4.2.2 THEOREM. *Assume that* $\mathbf{O}$ *is a coherent sheaf of left Noetherian rings and that* A *is a coherent* $\mathbf{O}$*-module. Then the homomorphisms (4.2.8) are isomorphisms for all* B *and* p.

Finally we note the following trivial case in which the homomorphisms (4.2.8) are isomorphisms, the two terms being 0:

4.2.3 PROPOSITION. *Assume that* A *is locally isomorphic to* $\mathbf{O}^n$ *(where* n *is a given finite positive integer). Then* $\mathbf{Ext}^p_{\mathbf{O}}(A, B) = 0$ *for all* $p > 0$ *and all* $\mathbf{O}$*-modules* B.

In fact, because of the local nature of $\mathbf{Ext}^p_{\mathbf{O}}(A, B)$, we can assume that $A = \mathbf{O}^n$, but then $\mathbf{Hom}_{\mathbf{O}}(A, B) = B^n$ is an *exact* functor in B, whence the conclusion.

COROLLARY. *If* A *is locally isomorphic to* $\mathbf{O}^n$*, we have*

$$\mathrm{Ext}^p_{\mathbf{O}}(X; A, B) = H^p(X, \mathbf{Hom}_{\mathbf{O}}(A, B))$$

for any $\mathbf{O}$*-module* B.

In fact, this is an immediate result of the spectral sequence of Theorem 4.2.1. In particular, if we take $A = \mathbf{O}$, we find:

$$H^p(X, B) = \mathrm{Ext}^p_{\mathbf{O}}(X; \mathbf{O}, B) \tag{4.2.9}$$

which is valid without any assumption on $\mathbf{O}$ or the $\mathbf{O}$-module B, and which is, moreover immediately trivial, since the $H^p(X, B)$ are derived functors, *in* $\mathbf{C}^{\mathbf{O}}$ (thanks to Proposition 4.1.3) of the functor $\Gamma(B) = \mathrm{Hom}_{\mathbf{O}}(\mathbf{O}, B)$.

To complete this section, we describe a method for calculating the spectral sequence of Theorem 4.2.1 which is more convenient than the one that would result from simply applying the definition.

4.2.4 PROPOSITION. *Let* $\mathbf{L} = (L_i)$ *be a left resolution of* A *by locally finitely free* $\mathbf{O}$*-modules such that* L_i *is locally isomorphic to* $\mathbf{O}^{n_i}$*. Consider the functor* $\mathbf{h_L}$ *on* $\mathbf{C}^{\mathbf{O}}$*, with values in the complexes of positive degree in* $\mathbf{C}$*, defined by* $\mathbf{h_L}(B) = \mathbf{Hom}_{\mathbf{O}}(\mathbf{L}, B)$*. We similarly set* $\mathbf{h}_A(B) = \mathbf{Hom}_{\mathbf{O}}(A, B)$*. Then* $\mathbf{h_L}$ *resolves the functor* $\mathbf{h}_A$ *(cf. 2.5).*

In fact, $\mathbf{h_L}$ is an exact functor (Proposition 4.2.3); moreover, $H^0(\mathbf{h_L}(B)) = \mathbf{h}_A(B)$, given that the functor $C \mapsto \mathbf{Hom}_{\mathbf{O}}(C, B)$ is left exact in C; and if B is injective, $\mathbf{h_L}(B)$ is an acyclic complex in positive dimensions, since in that case, the functor $C \mapsto \mathbf{Hom}_{\mathbf{O}}(C, B)$ is exact.

Corollary 1. *Under the conditions of the preceding proposition, we have*

$$\mathbf{Ext}^n(A,B) = H^n(\mathbf{Hom_O}(\mathbf{L},B)) \tag{4.2.10}$$

whence

$$\mathbf{Ext}^n(A,B)(x) = \mathrm{Ext}^n_{\mathbf{O}(x)}(A(x),B(x)) \tag{4.2.10$'$}$$

In fact, it is sufficient to apply Proposition 2.5.1.

Applying Proposition 2.5.4, we get:

Corollary 2. *Under the preceding conditions, the spectral functor of Theorem 4.2.1 is isomorphic to the spectral functor* $\mathrm{II}\,\Gamma(\mathbf{Hom_O}(\mathbf{L},B))$*; in particular, we have*

$$\mathrm{Ext}^n_{\mathbf{O}}(X;A,B) = \mathcal{R}^n\Gamma\mathbf{Hom_O}(\mathbf{L},B)$$

where the second term is the nth hyperhomology group of Γ *computed from the complex* $\mathbf{Hom_O}(\mathbf{L},B)$ *(cf. 2.4).*

Since we also have available a resolving functor for Γ, for example, the functor $\Gamma\mathbf{C}(F)$ where $\mathbf{C}(F)$[z] is the *canonical resolution* of F (cf. 2.3), there is an explicit spectral sequence for $\mathrm{II}\,\Gamma(\mathbf{Hom_O}(\mathbf{L},B))$: namely, the first spectral sequence of the bicomplex $\Gamma\mathbf{C}(\mathbf{Hom_O}(\mathbf{L},B))$, where we take as the first degree term the one that comes from $\mathbf{C}$. Now we have a natural equivalence

$$\mathbf{C}(\mathbf{Hom_O}(\mathbf{L},B)) \cong \mathbf{Hom_O}(\mathbf{L},\mathbf{C}(B)) \tag{4.2.11}$$

In fact, for any $\mathbf{O}$-module L there is a natural transformation

$$\mathbf{C}(\mathbf{Hom_O}(L,B)) \longrightarrow \mathbf{Hom_O}(L,\mathbf{C}(B)) \tag{4.2.12}$$

(where in the second term, $\mathbf{C}(B)$ is an $\mathbf{O}$-module in the obvious way), defined recursively on the dimension of the components of $\mathbf{C}$, starting with the natural transformation $\mathbf{C^O}(\mathbf{Hom_O}(L,B)) \longrightarrow \mathbf{Hom_O}(L,\mathbf{C^O}(B))$ derived from the natural transformations

$$\mathbf{Hom_O}(L,B)(x) \longrightarrow \mathrm{Hom}_{\mathbf{O}(x)}(L(x),B(x))$$

The latter is an isomorphism when L is pseudo-coherent (Proposition 4.1.1), from which we conclude that in this case, (4.2.12) is an isomorphism, which establishes, in particular, formula (4.2.11). From this we get $\Gamma\mathbf{C}(\mathbf{Hom_O}(\mathbf{L},B)) \cong \mathrm{Hom}_{\mathbf{O}}(\mathbf{L},\mathbf{C}(B))$, whence:

[z] Translator's note: Warning: the **C** here and elsewhere is not the category **C**; apparently it is in bold because it is a sheaf. It would have helped to have chosen a font for categories and another one for sheaves.

Corollary 3. *With* $\mathbf{L}$ *as in Proposition 4.2.1, the spectral functor of Theorem 4.2.1 is given by the first spectral sequence* $\mathrm{I}\,\mathrm{Hom}_{\mathbf{O}}(\mathbf{L}, \mathbf{C}(B))$ *of the bicomplex* $\mathrm{Hom}_{\mathbf{O}}(\mathbf{L}, \mathbf{C}(B))$*, where* $\mathbf{C}(B)$ *is a canonical resolution (cf. 2.3) of the abelian sheaf* B*, whose grading is taken as the first degree of a bicomplex. In particular, we have*

$$\mathrm{Ext}^n_{\mathbf{O}}(X; A, B) = H^n(\mathrm{Hom}_{\mathbf{O}}(\mathbf{L}, \mathbf{C}(B)))$$

Remark 1. We can also define the functors $\mathrm{Ext}^p_{\mathbf{O},\Phi}(X; A, B)$ as the right derived functors with respect to B, of the functor $\mathrm{Hom}_{\mathbf{O},\Phi}(A, B) = \Gamma(\mathbf{Hom}_{\mathbf{O}}(A, B))$ (where Φ is an cofilter of closed subsets of X). The preceding considerations are still valid. We will not have to use them.

Remark 2. In Theorem 4.2.2, it is essential that A satisfy a finiteness condition. For example, if $A = \mathbf{O}^{(I)}$ where I is infinite, then the second term of (4.2.8) vanishes for every X and every $p > 0$; however we will in general *not* have $\mathbf{Ext}^1_{\mathbf{O}}(A, B) = 0$, since the functor $B \mapsto \mathbf{Hom}_{\mathbf{O}}(\mathbf{O}^{(I)}, B) = B^I$ is generally not an exact functor.

4.3 Case of a constant sheaf of rings

Assume that $\mathbf{O}$ is a constant sheaf of rings defined by a unital ring O. Let M be an O-module, and let $\mathbf{M}$ be the constant $\mathbf{O}$-module it defines. Then for any $\mathbf{O}$-module A, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{O}}(\mathbf{M}, A) \cong \mathrm{Hom}_O(M, \Gamma(A))) \tag{4.3.1}$$

This is a natural equivalence that can also be written

$$h_{\mathbf{M}} = h_M \Gamma \tag{4.3.2}$$

in which we set, as we did in the preceding section, $h_{\mathbf{M}}(A) = \mathrm{Hom}_{\mathbf{O}}(\mathbf{M}, A)$ and $h_M(N) = \mathrm{Hom}_O(M, N)$ for any $\mathbf{O}$-module A and any O-module N. Γ is considered to be a functor from $\mathbf{C}^{\mathbf{O}}$ to $\mathbf{C}^O$ of left O-modules, and h_M to be a functor from $\mathbf{C}^O$ to the category of abelian groups. Theorem 3.4.1 applies, thanks to

4.3.1 Lemma. *If* A *is an injective* $\mathbf{O}$*-module, then* $\Gamma(A)$ *is an injective* O*-module.*

It is necessary to show that the functor $M \mapsto \mathrm{Hom}_{\mathbf{O}}(M, \Gamma(A))$ is exact, which results immediately form formula (4.3.1), since $M \mapsto \mathbf{M}$ is an exact functor. Using Theorem 3.4.1, and noting that the derived functors of Γ are the same, whether we consider Γ as taking its values in $\mathbf{C}^O$ or in the category $\mathbf{C}$ of abelian groups, we get

4.3.2 Theorem. *Let X be a space, O be a unital ring, M be an O-module, and $\mathbf{O}$ and $\mathbf{M}$ be the constant sheaves on X defined by O and M, respectively. Then there exists a cohomological spectral functor on $\mathbf{C}^{\mathbf{O}}$, abutting on the graded functor $(\mathrm{Ext}^p_{\mathbf{O}}(X;\mathbf{M},A))$, whose initial term is*

$$\mathrm{II}_2^{p,q}(A) = \mathrm{Ext}^q_{\mathbf{O}}(M, H^p(X,A)) \tag{4.3.3}$$

We derive in the usual way edge homomorphisms and a five term exact sequence, which we leave to the reader. To calculate the spectral functor, we use Proposition 2.5.3, Corollary 1, using the resolving functor $\Gamma\mathbf{C}(A)$ of Γ (which is considered here to be a covariant functor from $\mathbf{C}^{\mathbf{O}}$ to $\mathbf{C}^{O}$), $\mathbf{C}(A)$ being the canonical resolution of A (cf. 2.3), and the resolving functor h_L of h_M, where L is a left resolution of M by free O-modules. We find the first part of

4.3.3 Proposition. *Using the conditions of Theorem 4.3.1, we choose a left resolution L of M by free O-modules. Then the spectral functor of Theorem 4.3.1 is calculated by taking the second spectral sequence of the bicomplex $\mathrm{Hom}_O(L, \Gamma\mathbf{C}(A))$, where $\mathbf{C}(A)$ is the canonical resolution of A (cf. 2.3). If we assume that for each i, L_i is isomorphic to O^{n_i} (n_i being a positive integer), the first spectral sequence of this bicomplex gives the spectral functor of Theorem 4.2.1.*

For the latter statement, it suffices to consider the resolution $\mathbf{L}(\mathbf{M})$ derived from L and to apply to it Proposition 4.2.4, Corollary 3. The preceding proposition reduces the calculation of $\mathrm{Ext}_{\mathbf{O}}(\mathbf{M}, A)$ and its spectral sequences to a classical problem of the Künneth type. Thus:

Corollary 1. *Assume that A is annihilated by a 2-sided ideal $I \subseteq O$ such that O/I is the semisimple ring K. Then both spectral sequences of the bicomplex $\mathrm{Hom}_O(L, \Gamma\mathbf{C}(A))$ are trivial, and their initial terms can be canonically identified with the homology $\mathrm{Ext}_{\mathbf{O}}(X;\mathbf{M},A)$ of this bicomplex. More precisely, we have canonical isomorphisms:*

$$\mathrm{Ext}^n_{\mathbf{O}}(X,\mathbf{M},A) \cong \bigoplus_{p+q=n} \mathrm{Hom}_K(\mathrm{Tor}^O_p(K,M), H^q(X,A)) \tag{4.3.4}$$

$\mathbf{C}(A)$ and therefore $\Gamma\mathbf{C}(A)$ will thus be annihilated by I, from which we conclude

$$\mathrm{Hom}_O(L, \Gamma\mathbf{C}(A)) = \mathrm{Hom}_K(K \otimes_O L, \Gamma\mathbf{C}(A)) \tag{4.3.5}$$

and since K is semisimple, we can apply the simplest version of the Künneth theorem (owing to the fact that Hom_K is an exact bifunctor). We find the triviality of the spectral sequences, and in addition, the formula $H(\mathrm{Hom}_K(K\otimes_O L, \Gamma\mathbf{C}(A))) = \mathrm{Hom}_K(H(K\otimes_O L), H(\Gamma\mathbf{C}(A)))$, which is exactly formula (4.3.4).

We continue to assume that A is annihilated by a 2-sided ideal I but we make no supposition about $K = O/I$. We have (4.3.5), which leads us to consider the hyperhomology spectral sequences of the bifunctor Hom_K applied to the complexes $K \otimes_O L$ and $\Gamma\mathbf{C}(A)$. In the first, the initial term $H^p(\mathrm{Ext}^q_K(K \otimes_O L, \Gamma\mathbf{C}(A))) = 0$ for $q > 0$, since $K \otimes_O L$ is K-free. Thus the abutment is canonically identified with $H(\mathrm{Hom}_K(K \otimes_O L, \Gamma\mathbf{C}(A)))$, which is what we must calculate. In the second spectral sequence, the initial term is $\bigoplus_{q'+q''=q} \mathrm{Ext}^p_K(H^{q'}(K \otimes_O L), H^{q''}\Gamma\mathbf{C}(A))$. Therefore:

Corollary 2. *Assume that the* $\mathbf{O}$*-module* A *is annihilated by a 2-sided ideal* I *of* $\mathbf{O}$ *and set* $K = \mathbf{O}/I$. *Then* $\mathrm{Ext}^n_{\mathbf{O}}(X; \mathbf{M}, A)$ *is also the abutment of a third spectral functor, whose initial term is*

$$\mathrm{III}^{p,q}_2(A) = \bigoplus_{q'+q''=q} \mathrm{Ext}^p_K(Tor^O_{q'}(K, M), H^{q''}(X, A)) \tag{4.3.6}$$

In particular, if K *is a "left hereditary ring" [6, 1.5] (for example, a PIR), we have a canonical exact sequence:*

$$0 \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Ext}^1_K(\mathrm{Tor}^O_p(K, M), H^q(X, A)) \longrightarrow \mathrm{Ext}^n_{\mathbf{O}}(X; \mathbf{M}, A)$$
$$\longrightarrow \bigoplus_{p+q=n} \mathrm{Hom}_K(\mathrm{Tor}^O_p(K, M), H^q(X, A)) \longrightarrow 0$$

Remark 1. Assume that $L_i = O^{n_i}$, for all i. Then (Proposition 4.3.2) the first hyperhomology spectral sequence of the functor $\mathrm{Hom}_O(M, -)$ for the complex $\Gamma\mathbf{C}(A)$ is the sequence in Theorem 4.3.1, whose initial term thus has the significant interpretation (4.2.5). But we must pay close attention that this fact was inferred from Corollary 3 of Proposition 4.2.4; it is basically dependent on Formula (4.2.11), and therefore *on the fact that the "canonical resolution" functor* $\mathbf{C}$ *commutes with the functor* $\mathrm{Hom}_{\mathbf{O}}(\mathbf{L}, -)$ if L is an O-module isomorphic to O^n (thanks to which one is moreover in the context of the corollary to Proposition 2.5.2). However, this is still true (for other reasons) if we replace $\mathbf{C}(A)$ by the "Cartan resolution" $A \otimes ZF$ (where F is a "fundamental sheaf" on a paracompact space X).

Remark 2. Let $x \in X$. We have seen in 4.2 that we have

$$\mathbf{Ext}^p_{\mathbf{O}}(\mathbf{M}, A)(x) = \varinjlim \mathrm{Ext}^p_{\mathbf{O}}(U; \mathbf{M}, A)$$

the inductive limit taken over the filter of open neighborhoods U of x. For any U, we can write the spectral sequences II and III, from which we conclude, by an easy passage to the limit, that $(\mathbf{Ext}^p_{\mathbf{O}}(\mathbf{M}, A)(x))_p$ *is the abutment of two spectral sequences*, whose initial terms are, respectively,

$$\mathrm{II}^{p,q}_2 = \varinjlim \mathrm{Ext}^p_O(M, H^q(U, A))$$

$$\mathrm{III}_2^{p,q} = \bigoplus_{q'+q''=q} \varinjlim \mathrm{Ext}_K^p(\mathrm{Tor}_{q'}^k(K,M), H^{q''}(U,A))$$

(In the second spectral sequence, we assume that A is annihilated by a 2-sided ideal I and $K = O/I$.) The result is that whenever, in one of these spectral sequences, it is possible to transfer the symbol $\varinjlim$ on $H(U,A)$ into each of the components of the initial term, then the formula $\mathbf{Ext}^p_{\mathbf{O}}(\mathbf{M},A)(x) = \mathrm{Ext}^p_O(M,A)(x)$ holds (its validity is assured, a priori, only if M admits a projective resolution by modules O^{n_i} of finite type). Here are two typical cases where this happens: (a) A is the *constant* sheaf defined by an O-module N and X is locally compact Hausdorff (we look at the first spectral sequence); (b) A is annihilated by an ideal I such that $K = O/I$ is Noetherian, and the $\mathrm{Tor}^O_q(K,M)$ are of finite type over K. Now assume that A is the constant sheaf defined by an O-module N annihilated by I and, to simplify, assume that $K = O/I$ is a field. Then $\mathbf{Ext}^n_{\mathbf{O}}(\mathbf{M},\mathbf{N})$ is the sheaf associated to the presheaf defined by the $\bigoplus_{p+q=n} \mathrm{Hom}_K(\mathrm{Tor}^O_p(K,M), H^q(U,M))$. It is easy to construct examples (with $M = N = K$, $p = 1$, O the group algebra $K(G)$ of a group G, and I the augmentation ideal) where this sheaf is distinct from the constant sheaf associated to $\mathrm{Ext}^p_O(M,N)$ (which is $H^p(G,K)$); in this case, the conclusion of Theorem 4.2.2 fails.

4.4 Case of sheaves with an operator group

Let G be a group and k be a commutative ring with unit. We set $O = k(G)$ (the group algebra of G with coefficients in k); O is an augmented algebra whose augmentation ideal will be denoted by I. Therefore k can be identified with O/I. An $\mathbf{O}$-module A is then a sheaf of k-modules admitting G as an operator group. Saying that I annihilates A means that G operates trivially (as identity operators) on A. Because of their significance for the following chapter, we will quickly review the essential notation and results from the preceding sections in the present case. Omitting mention of k in what follows (in practice k will be the ring $\mathbf{Z}$ of integers or a field)[aa], we will say G-module or G-sheaf instead of $\mathbf{O}$-module or $\mathbf{O}$-sheaf. If A and B are two G-sheaves, we will write $\mathrm{Hom}_G(B,A)$, $\mathbf{Hom}_G(B,A)$, $\mathrm{Ext}^p_G(X;B,A)$, and $\mathbf{Ext}^p_G(B,A)$ (using G as subscript instead of $\mathbf{O}$). In the case that $B = k$ (the only significant one in what follows), the preceding objects will also be denoted $\Gamma^G(A)$, A^G, $H^p(X;G,A)$ and $\mathbf{H}^p(G,A)$. Thus Γ^G is the functor h_k; using the preceding notation, we have $\Gamma^G(A) = \Gamma(A)^G$ (the group of G-invariant elements of $\Gamma(A)$) $= \Gamma(A^G)$, where A^G denotes the subsheaf of A consisting of the germs of G-invariant sections (if G admits a *finite* system of generators, then A^G is also the set of G-invariant elements of A). The $\mathbf{H}^p(G,A)$, $(-\infty < p < +\infty)$ are functorial sheaves in A, which form the cohomological functor derived from $\Gamma^G(A) = A^G$. We have canonical homomorphisms

$$\mathbf{H}^p(G,A)(x) \longrightarrow H^p(G,A(x))$$

[aa] Translator's note: In light of this remark, we have changed a number of Z in this section to $\mathbf{Z}$.

which are bijective in all cases we will consider, and always when G is finite (by virtue of either Theorem 4.2.2 or Corollary 1 of Proposition 4.2.4). These functors are of local nature, that is they commute with the operations of restriction to open sets. The $H^p(X; G, A)$ form the cohomological functor derived from $\Gamma^G(A)$. We have:

4.4.1 THEOREM. *There are two cohomological spectral functors on the category of G-sheaves abutting on the graded functor $(H^n(X; G, A))$, whose initial terms are, respectively,*

(4.4.1)
$$\mathrm{I}_2^{p,q}(A) = H^p(X, H^q(G, A))$$
$$\mathrm{II}_2^{p,q} = H^p(G, H^q(X, A))$$

These are also the two spectral sequences of the "operator complex" $\mathbf{C}(A)$, the canonical resolution of A, where the two hypercohohomology spectral sequences of X with respect to the complex $C(G, A)$ are "cochains on G with coefficients in the sheaf A". These two spectral sequences are trivial if G operates trivially on A and if, in addition, A is a vector space over the field k. We then have a canonical isomorphism:

$$H^n(X; G, A) \cong \bigoplus_{p+q=n} \mathrm{Hom}_k(H_p(G, k), H^q(X, A)) \tag{4.4.2}$$

More generally, assume only that G acts trivially on A. Then Corollary 2 to Proposition 4.3.2 gives a canonical exact sequence

$$\begin{aligned} 0 \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Ext}^1_{\mathbf{Z}}(H_p(G, \mathbf{Z}), H^q(X, A)) \longrightarrow H^n(X; G, A) \\ \longrightarrow \bigoplus_{p+q=n} \mathrm{Hom}_{\mathbf{Z}}(H_p(G, \mathbf{Z}), H^q(X, A)) \longrightarrow 0 \end{aligned} \tag{4.4.3}$$

REMARK 1. In this last formula, we presupposed that $k = \mathbf{Z}$. However this is unimportant, since if k is arbitrary and A is a sheaf of k-modules on which G operates, then the $\mathbf{H}^p(G, A)$ and $H^p(X; G, A)$ are the same, whether A is thought of as a sheaf of $k(G)$-modules or a sheaf of $\mathbf{Z}(G)$-modules. This reduces to proving that if A is an injective sheaf of $k(G)$-modules, then $\mathrm{Ext}^n_{\mathbf{Z}(G)}(X; \mathbf{Z}, A)$ and $\mathbf{Ext}^n_{\mathbf{Z}(G)}(\mathbf{Z}, A)$ (where we write $\mathbf{Z}$ and $\mathbf{Z}(G)$ for the constant sheaves they represent) vanish for $n > 0$. For the first claim, this results from the spectral sequence II, which shows that it is isomorphic to $H^n(G, \Gamma(A))$ which vanishes since $\Gamma(A)$ is an injective k-module (for which the result referred to is well known). Since $\mathbf{Ext}^n_{\mathbf{Z}(G)}(\mathbf{Z}, A)$ is the sheaf associated to the presheaf of the $\mathrm{Ext}^n_{\mathbf{Z}(G)}(U; \mathbf{Z}, A)$, which vanishes according to the preceding (since $A|U$ is also injective), our assertion follows.

REMARK 2. Assume that A is the *constant* sheaf defined by a G-module M. Then we can obviously write $H^p(X; G, M)$ and $\mathbf{H}^p(G, M)$ instead of $H^p(X; G, A)$ and $\mathbf{H}^p(G, A)$. If G acts trivially on M, then $H^p(X; G, M)$ can be calculated completely using formula (4.4.3), or formula (4.4.2) if M is a vector space over a field k. In the latter case, if X is a locally compact Hausdorff space, we even get: $H^*(X; G, M)$ is the space of bilinear functions from $H_*(G, k) \times H_*(X, k)$ to M. We should note that in general, $\mathbf{H}^p(G, M)$ is not the constant sheaf defined by $H^p(G, M)$; cf. Remark 2 of 4.3.

Chapter 5

Cohomological study of operator spaces

5.1 Generalities on G-sheaves

In this entire chapter, we will be considering a space X on which a group G acts (on the left, say). The operation defined by $g \in G$ will be denoted $x \mapsto g \cdot x$. We do not require G to act faithfully; in particular, we will also be considering the case in which G acts trivially. We will $X(G)$ for X equipped with the additional structure of a G-action. The orbit space X/G will be denoted Y; it will be equipped with the quotient topology. The canonical projection of X onto Y will be denoted f; it is an open continuous function. Y will be considered to be equipped with trivial action from G; we will therefore write $Y(G)$ as a reminder of this additional structure on Y.

We will call a *G-sheaf* on $X = X(G)$ a sheaf (of sets) A over X, on which G acts in a way that is compatible with its action on X. To explain this definition, we could, for example, consider A to be the etale space over X (cf. 3.1); we will not pursue this point of view further. Similarly we define the notion of *G-sheaf of groups, and of rings, as well as abelian G-sheaves* (= G-sheaf of abelian groups), etc. In practical terms we can say that if X is equipped with certain structures and if a sheaf A over X is defined in structural terms, then A naturally inherits the structure of a G-sheaf if G acts as a group of automorphisms of X. Thus a constant sheaf can always be considered a G-sheaf (*"trivial G-sheaf"*); similarly for the sheaf of germs of arbitrary (respectively continuous) functions from X to a given set; and similarly for the sheaf of germs of holomorphic functions when X is a holomorphic variety and the actions of G are holomorphic automorphisms of X; etc. We call *G-homomorphism* from one sheaf to another a sheaf homomorphism that commutes with the G-action. If the sheaves in question are sheaves of groups, for example, we presuppose that the homomorphism respects that structure as usual. With this notion of homomorphism, the G-sheaves of sets (respectively groups, etc.) form a *category* (cf. 1.1), which has the same properties as the corresponding category without a group of operators (which moreover is a special case of it, corresponding to the case in which $G = 1$).

In particular, the abelian G-sheaves from an additive category, whose properties we will describe. More generally, let $\mathbf{O}$ be a G-sheaf of unital rings. We consider the sheaves A over X, which are both abelian G-sheaves and $\mathbf{O}$-modules. The operations of $\mathbf{O}$ on A are compatible with the operations of $\mathbf{O}$ (i.e., the natural homomorphisms of set-valued sheaves $\mathbf{O} \times A \longrightarrow A$ are G-homomorphisms). Such a sheaf will be called a G-$\mathbf{O}$-*module*. Calling a homomorphism of sheaves that is both a G-homomorphism and an $\mathbf{O}$-homomorphism a G-$\mathbf{O}$-*homomorphism*, we see that the sum or composite of two G-$\mathbf{O}$-homomorphisms is again a G-$\mathbf{O}$-homomorphism, and thus the G-$\mathbf{O}$-modules form an *additive category*, denoted $\mathbf{C}^{\mathbf{O}(G)}$. If $\mathbf{O}$ is the constant sheaf of integers (with the trivial G-actions), we obtain once more the category of abelian G-sheaves, denoted $\mathbf{C}^{X(G)}$. If G operates trivially on X, the category $\mathbf{C}^{\mathbf{O}(G)}$ can be interpreted as the category of $\mathbf{O}'$-modules, where $\mathbf{O}'$ is an appropriate sheaf of rings. If, for example, G operates trivially on $\mathbf{O}$, we have $\mathbf{O}' = \mathbf{O} \otimes_{\mathbf{Z}} \mathbf{Z}(G)$ (where $\mathbf{Z}(G)$ denotes the integral group algebra of G, or rather the constant sheaf that it defines). The results of 3.1 remain valid with minor modifications:

5.1.1 PROPOSITION. *Let $\mathbf{O}$ be a G-sheaf of rings on X. Then the additive category $\mathbf{C}^{\mathbf{O}(G)}$ of G-$\mathbf{O}$-modules is an abelian category satisfying axioms* AB 5) *and* AB 3*) *of 1.5, and admits a generator.*[9]

The proof is trivial, except for the existence of a generator, for which we now give a construction that generalizes the one in 3.1. For every open set $U \subseteq X$, let $L(U)$ be the $\mathbf{O}$-module that is the direct sum of the $\mathbf{O}$-modules $\mathbf{O}_{g\cdot U}$ for $g \in G$. $L(U)$ can clearly be considered a G-sheaf, such that $L(U)$ even becomes a G-$\mathbf{O}$-module. If A is an arbitrary G-$\mathbf{O}$-module, a $\mathbf{G}$-$\mathbf{O}$-homomorphism from $L(U)$ to A is known when we know its restriction to $\mathbf{O}_U$, which is an $\mathbf{O}$-homomorphism from $\mathbf{O}_U$ to A, which can be arbitrarily specified in advance. Providing such a homomorphism is also equivalent to providing a section from A to U. If B is a G-$\mathbf{O}$-submodule of A, distinct from A, then on an appropriate open set U we can find a section of A that is not a section of B. Consequently, the family of $L(U)$ is a family of generators of $\mathbf{C}^{\mathbf{O}(G)}$, and their direct sum (for U ranging over the open sets of X) is therefore a generator.

It follows in particular from Proposition 4.1 that every G-$\mathbf{O}$-sheaf is isomorphic to a

[9] This proposition, as well as various subsequent results can also be stated in the following more general context. We take a category $\mathbf{C}$, a group G, and a "representation of G by functors in $\mathbf{C}$": a functor $F_g : \mathbf{C} \longrightarrow \mathbf{C}$ is associated to each element $g \in G$ such that we have F_e is the identity functor and $F_g F_{g'} = F_{gg'}$ (up to *given* natural transformations, satisfying certain coherence conditions that we leave to the reader). Then we can construct the category $\mathbf{C}^G$ of "objects of $\mathbf{C}$ with operator group G", just as $\mathbf{C}^{X(G)}$ (for a space with operator group $X(G)$) can be constructed using $\mathbf{C}^X$ (compare also with the special case, Example 1.7.f). Many properties that are true for $\mathbf{C}$ are inherited by $\mathbf{C}^G$.

subsheaf of an injective $G-\mathbf{O}$-module. It is important for what follows to obtain a convenient explicit form for "sufficiently many" injective objects, by generalizing the construction in 3.1. Let $(A_x)_{x\in X}$ be a family of $\mathbf{O}(x)$-modules. Consider the $\mathbf{O}$-module that it defines (cf. 3.1). We define an operation of G on the product so that we get a $G-\mathbf{O}$-sheaf. To do so, if suffices for every $x \in X$ and $g \in G$ to provide a homomorphism, denoted $A \longrightarrow g \cdot A$, from the abelian group A_x to the abelian group $A_{g\cdot x}$ such that $g\cdot(ua) = (g\cdot u)(g\cdot a)$ for $u \in \mathbf{O}(X)$ and $a \in A_x$, and such that $e\cdot a = a$ and $g\cdot(g'\cdot a) = (gg')\cdot a$. We now introduce the auxiliary ring U_x generated by the group algebra $\mathbf{Z}(G_x)$ of G_x and the ring $\mathbf{O}(x)$, subject to the commutative relations $gug^{-1} = u^g$ for $u \in \mathbf{O}(x)$, $g \in G_x$ (where, to avoid confusion, we denote by u^g the conjugate of u by $g \in G_x$). We see that the A_x are in fact U_x-modules and any $g \in G$ defines an isomorphism $a \mapsto g\cdot a$ from the group A_x to $A_{g\cdot x}$ that satisfies $g\cdot(ua) = (g\cdot u)(g\cdot a)$ for $u \in U_x$, $a \in A_x$. It easily follows that, if we choose an element $\xi(y)$ in any orbit $y \in Y$ and if we set $U_y = U_{\xi(y)}$, the preceding data are equivalent to providing a family $(A_y)_{y\in Y}$ of U_y-modules A_y (which is equal to $A_{\xi(y)}$).

The A_x $(x \in y)$ can be deduced from A_y as follows. Let A_y be the G-module *induced* by the G_y-module A_y, i.e. the G-module $\mathrm{Hom}_{G_y}(\mathbf{Z}(G), A_y)$ obtained from the G_y-module A_y by "contravariant scalar extension": A_y is then identified with a quotient group of $\overline{A}_y$ by a stable abelian subgroup of G_y, and $\overline{A}_y$ is identified with the product of all the conjugate quotients $g\cdot A_y$ for $g \in G/G_y$. We similarly introduce the G-module $\overline{O}_y$ induced by $\mathbf{O}_y = \mathbf{O}(\xi(y))$, and we immediately observe that $\overline{A}_y$ is an $\overline{O}_y$-module identified with the product of the $g\cdot O_y$-modules $g\cdot A_y$ for $g \in G/G_y$. We then have the canonical isomorphisms $g\cdot\mathbf{O}_y \cong \mathbf{O}(g\cdot\xi(y))$ and $g\cdot A_y \cong A_{g\cdot\xi(y)}$. We denote by $P(A)$ the $G-\mathbf{O}$-sheaf on X defined by the family $A = (A_y)$ of U_y-modules A_y. Let B be any $G-\mathbf{O}$-sheaf. A $\mathbf{O}$-homomorphism from a $G-\mathbf{O}$-module B to $P(A)$ is identified with a family $(v_x)_{x\in X}$ of $\mathbf{O}(x)$-homomorphisms $B(x) \longrightarrow A_x$ (as we indicated in 3.1), and this $\mathbf{O}$-homomorphism is compatible with G if and only if the v_x "commute" with G. We immediately see that it comes to the same thing as taking a $G-\mathbf{O}$-homomorphism from B to $P(A)$, or a family $(v_y)_{y\in Y}$ of U_y-homomorphisms $B(\xi(y)) \longrightarrow A_y$. This leads to:

5.1.2 Proposition. *Let $(A_y)_{y\in Y}$ be a family of injective U_y-modules. Then the $G-$ $\mathbf{O}$module $P((A_y))$ that it defines is injective. Moreover, any $G-\mathbf{O}$-module is isomorphic to a subsheaf of a sheaf of the preceding type.*

This last fact (which is the most important for us) is immediate when we embed, for any y, the module in question $B(\xi(y))$ from the given sheaf B, into an injective U_y-module.

Direct images of G-sheaves. Let A be a sheaf of sets over X. Recall that we defined its direct image $f_*(A)$ as the sheaf over Y whose sections on the open set $U \subseteq Y$ are the sections of A over $f^{-1}(U)$. If A is a G-sheaf, then $\Gamma(f^{-1}(U), A)$ admits G as a group of operators from which we easily conclude that *the direct image $f_*(A)$ of a G-sheaf over X is a G-sheaf over Y* (recall that G operates trivially on Y). Of course, if A is a G-sheaf of groups, or of rings, etc., the same is true of $f_*(A)$. If A is a G-sheaf, we denote by A^G or $f_*^G(A)$ the sheaf Γ^G of $f_*(A)$ of the invariants of the G-sheaf $f_*(A)$ over Y, in other words, the sheaf whose sections over an open set $U \subseteq Y$ are the sections of A over $f^{-1}(U)$ *that are invariant under* G. Clearly, if A is a G-sheaf of groups or of rings, etc., this is also true for $A^G = f_*^G(A)$. Moreover, f_*^G can be considered a covariant *functor* defined over the category of G-sheaves on X, with values in the category of sheaves on Y (considered as G-sheaves on which G operates trivially). If $\mathbf{O}$ is a G-sheaf of rings and if we denote by $\mathbf{O'}$ the sheaf $f_*^G(\mathbf{O})$, then for any $G-\mathbf{O}$-module A over X, $f_*(A)$ is a G-$\mathbf{O'}$-module over Y. Thus f_* is a covariant *functor* from $\mathbf{C}^{\mathbf{O}(G)}$ to $\mathbf{C}^{\mathbf{O'}(G)}$ which is additive and left exact, as usual. f_*^G is essentially the composite functor $\Gamma^G f_*$ from $\mathbf{C}^{\mathbf{O}(G)}$ to the category $\mathbf{C}^{\mathbf{O'}}$ of $\mathbf{O'}$-modules on Y, with Γ^G denoting the functor $\mathbf{C}^{\mathbf{O'}(G)} \longrightarrow \mathbf{C}^{\mathbf{O'}}$ that associates to a G-$\mathbf{O'}$-sheaf B on Y the sheaf B^G of germs of invariant sections under G. To simplify, assume now that $\mathbf{O}$, and thus also $\mathbf{O'}$, is the constant sheaf $\mathbf{Z}$ of integers (which amounts to not considering sheaves of rings at all). Then we have:

5.1.3 Proposition. *f_* transforms injectives of $\mathbf{C}^{X(G)}$ into injectives of $\mathbf{C}^{Y(G)}$.*

In fact, it is sufficient to see this for an abelian G-sheaf A, defined by a family (A_y) of injective G_y-modules (Proposition 5.1.2). Reverting to the notation in the construction that preceded Proposition 5.1.2, we see that $f_*(A)$ is the product sheaf defined by the groups $\overline{A}_y = \prod_{x\in f\,\inf(y)} A_x$, and the G-sheaf structure on $f_*(A)$ is the one defined by the G-module structure on $\overline{A}_y$. Since the latter was obtained from the injective G_y-module A_y by contravariant extension of scalars, it too is clearly injective [6, II, Proposition 6.1.a]. Thus f_*A is injective by virtue of Proposition 3.1.2.

Corollary. *If A is an injective abelian G-sheaf, then $\Gamma(A)$ is an injective G-module, and A^G is a flabby sheaf on Y.*

In fact, we have $\Gamma(A) = \Gamma(f_*(A))$. $f_*^G(A) = \Gamma^G(f_*(A))$ and it then suffices to apply the lemma in 4.3, respectively Proposition 4.1.3 (with $O = \mathbf{Z}(G)$ and $\mathbf{O}$ the constant sheaf of rings defined by O).

Inverse images and direct images. Let B be a sheaf of sets on Y (without operators). Then *its inverse image* $f^{-1}(A)$ (mentioned in 3.2) can be thought of as a G-sheaf in a natural fashion, for obvious reasons of "structure transport". A section of B over an open set U defines, by inverse image, a section of $f^{-1}(B)$ over $f^{-1}(U)$ *invariant under* G, i.e. a section of $f_*^G(f^{-1}(B))$, whence a natural homomorphism $B \mapsto f_*^G(f^{-1}(B))$ and we can immediately show that there is an isomorphism:

$$f_*^G(f^{-1}(B)) \cong B \tag{5.1.1}$$

Conversely, we begin with a G-sheaf A over X and consider $f^{-1}(f_*^G(\ ))$: a section of the sheaf on an open set V is determined by a function $g(x)$ over V whose value at each $x \in V$

is an element of $f_*^G(A)(f(x)) = \varinjlim_{f(x)\in U} \Gamma(f^{-1}(U),A)^G$, and such that for all $x \in V$, there is an open neighborhood $V' \subseteq V$ of x and a section h of $f_*^G(A)$ over a neighborhood U' of $f(x)$ (i.e. an invariant section h of A over $f^{-1}(U')$) such that we have $g(x) = h(x)$ for $x \in V' \cap f^{-1}(U')$. We conclude from this that there is a canonical monomorphism[10]:

$$f^{-1}(f_*^G(A)) \longrightarrow A \tag{5.1.2}$$

identifying $f^{-1}(f_*^G(A))$ as a subsheaf of $A' \subseteq A$. It follows from (5.1.1) that we have $A' = A$ if and only if A is isomorphic to a G-sheaf of the form $f^{-1}(B)$. If this holds for every $x \in X$, the operations of G_x on $A(x)$ are trivial. Moreover the converse is true if G satisfies condition (D) below (cf. 5.3), as we easily see. *If in addition, the G_x are reduced to* e (i.e. if no non-identity element of G has a fixed point), *the functors f_*^G and f^{-1} establish inverse isomorphisms between the category of G-sheaves on X and the category of sheaves on Y.*

5.1.4 EXAMPLES OF DIRECT IMAGES OF G-SHEAVES If A is the constant sheaf over X defined by a set M on which G operates trivially, then A^G is the constant sheaf over Y defined by the same set M; but if G operates non-trivially on M, and if we are not in the "fixed-point free" case, then $f^G(M)$ is obviously no longer constant in general. Assume that condition (D) of 5.3 is satisfied. Let A be the sheaf of germs of functions from X into a set M. Then A^G is the sheaf of germs of functions from Y into M. Under this correspondence, if X is a differentiable variety (or a real or complex analytic space), and if A is a sheaf of germs of functions (with values in a space of the same kind), which are differentiable (respectively, real analytic, respectively, complex analytic), then A^G is a sheaf of the same kind. This last example is especially important, and the analogous statement is true (by definition, moreover, in the same way as the preceding ones are), in abstract algebraic geometry, and even for arithmetic varieties.

5.2 The functors $H^n(X;G,A)$ and $\mathbf{H}^n(G,A)$ and the fundamental spectral sequences.

To avoid confusion, we denote as Γ_X and Γ_Y the "section" functors defined for sheaves on X, respectively, Y, and we denote by Γ^G the functor $M \mapsto M^G$, so that a set M on which G operates gives rise to a set of G-invariants of M. Finally, if A is a G-sheaf on X, we have

$$\Gamma_X^G(A) = \Gamma_X(A)^G \tag{5.2.1}$$

[10] More generally, if f is a continuous function from a space X to a space Y, we have, for the sheaves A over X, natural transformations $f^{-1}(f_*(A)) \longrightarrow A$, that can be readily seen to be monomorphic (5.1.2) thanks to the injection $A^G \longrightarrow f_*(A)$.

thus by definition we have the formulas

$$\Gamma_X^G = \Gamma^G \Gamma_X = \Gamma_Y f_*^G \tag{5.2.2}$$

limiting ourselves to taking A in the category $\mathbf{C}^{X(G)}$ of abelian G-sheaves on X, Γ_X^G is a left exact additive functor from $\mathbf{C}^{X(G)}$ to the category $\mathbf{C}$ of abelian groups, and f_*^G is a left exact functor from $\mathbf{C}^{X(G)}$ to the category $\mathbf{C}^Y$ of abelian sheaves (without operators) on Y. Then we have

$$H^n(X; G, A) = R^n \Gamma_X^G(A) \tag{5.2.3}$$

$$\mathbf{H}^n(G, A) = R^n f_*^G(A) \tag{5.2.4}$$

Thus, for an abelian G-sheaf, the $H^n(X; G, A)$ are abelian groups, and the $\mathbf{H}^n(G, A)$ are sheaves on Y. Each of them form universal cohomological functors of A, reducing for $n = 0$ to $\Gamma_X^G(A)$, respectively A^G. If G operates trivially on X, we recover the notion introduced in 4.4. Again, we can easily see, as in Proposition 3.1.3, that if U is an open subset of Y and A is an *injective* abelian G-sheaf on X, then its restriction to $f^{-1}(U)$ is an injective abelian G-sheaf on that space. From this we conclude, as in 4.2:

5.2.1 PROPOSITION. *Let A be an abelian G-sheaf on X. Then for every open set $U \subseteq Y$, we have $\mathbf{H}^p(G, A)|U = \mathbf{H}^p(G, A|f^{-1}(U))$.*

This is why we permit ourselves to omit X in the notation $\mathbf{H}^p(G.A)$. We get a more explicit reduction of the calculation of $\mathbf{H}^p(G, A)$ using:

COROLLARY. *Assume that $f^{-1}(U)$ is a union of pairwise disjoint open sets $g \cdot V$ (where $g \in G/G_0$), where V is open set in X and G_0 is a subgroup of G such that $g_0 \cdot V = V$ for $g_0 \in G_0$. Then $\mathbf{H}^p(G, A)|U = \mathbf{H}^p(G_0, A|V)$ (making use of the natural identification $U = V/G_0$).*

By virtue of Proposition 2.5.1, we can assume that $U = Y$, whence $f^{-1}(U) = X$. We can immediately see that the category of G-sheaves on X is isomorphic to the category of G_0-sheaves on V, this isomorphism being compatible with the functors f_*^G and $f_*^{G_0}$. This immediately gives rise the desired formula. A more explicit computation will be given below (cf. formula (5.2.11) and Theorem 5.3.1).

Formulas (5.2.2) represent Γ_X^G as a functor composed in two different ways, and in both cases, Theorem 2.4.1 is applicable, thanks to the corollary to Proposition 5.1.3. We thereby get:

5.2.2 THEOREM. *There exists on the category $\mathbf{C}^{X(G)}$ of abelian G-sheaves on X two cohomological spectral functors abutting on the graded functor $(H^n(X;G,A))$, whose initial terms are, respectively:*

$$\begin{cases} \mathrm{I}_2^{p,q}(A) = H^p(Y, \mathbf{H}^q(X,A)) \\ \mathrm{II}_2^{p,q}(A) = \mathbf{H}^p(G, H^q(X,A)) \end{cases} \tag{5.2.5}$$

To establish the explicit form of $\mathrm{II}_2^{p,q}(A)$, it suffices to show once more that the derived functors of Γ_X, considered as a functor from $\mathbf{C}^{X(G)}$ to the category $\mathbf{C}(G)$ of G-modules, are really the $H^q(X.A)$. This follows immediately from the fact that every abelian G-sheaf can be embedded into a flabby abelian G-sheaf (as we have seen in the preceding section), which thus makes the $H^q(X,A)$ vanish for $q > 0$.

These spectral sequences give rise initially to important edge homomorphisms:

$$H^n(Y, A^G) \longrightarrow H^n(X;G,A) \longrightarrow H^0(Y, \mathbf{H}^n(G,A)) \tag{5.2.6}$$

$$H^n(G, H^0(X,A)) \longrightarrow H^n(X;G,A) \longrightarrow H^n(X,A)^G \tag{5.2.7}$$

The second arrow in (5.2.6) is immediate if we remark that by virtue of Lemma 3.7.2, $\mathbf{H}^n(G,A)$ is the sheaf over Y associated to the presheaf formed by the $H^n(f^{-1}(U);G,A)$. The composite of the first homomorphism on the first line with the second homomorphism of the second line is the homomorphism f^* associated to the natural injection (5.1.2) from $f^{-1}(A^G)$ to A (cf. 3.2).

The spectral sequences of Theorem 5.2.1 also define two five-term exact sequences:

$$\begin{aligned} 0 \longrightarrow H^1(Y, A^G) \longrightarrow H^1(X;G,A) \longrightarrow H^0(Y, \mathbf{H}^1(G,A)) \\ \longrightarrow H^2(Y, A^G) \longrightarrow H^2(X;G,A) \end{aligned} \tag{5.2.8}$$

$$\begin{aligned} 0 \longrightarrow H^1(G, \Gamma A) \longrightarrow H^1(X;G,A) \longrightarrow H^1(X,A)^G \\ \longrightarrow H^2(G, \Gamma A) \longrightarrow H^2(X;G,A) \end{aligned} \tag{5.2.9}$$

There is a third way to represent Γ_X^G as a composite functor, namely $\Gamma_X^G = \Gamma_Y^G f_*$, and Theorem 2.4.1 is still applicable by virtue of Proposition 5.1.3. Thus the graded functor $(H^n(X;G,A))$ is also the abutment of a third spectral sequence whose initial term is

$$E_2^{p,q} = H^p(Y;G, R^q f_*(A))$$

This spectral sequence defines the natural transformations

$$H^n(Y;G,f_*(A)) \longrightarrow H^n(X;G,A) \tag{5.2.10}$$

This spectral sequence is of limited interest since it appears that it is pathological in cases where it is not trivial, that is unless the conditions of the following proposition are satisfied:

5.2.3 Proposition. *Assume that there is an abelian G-sheaf A such that $R^q f_*(A) = 0$ for $q > 0$. Then the homomorphisms (5.2.10) are isomorphisms; moreover the two spectral sequences of Theorem 5.2.1 are identified with the two corresponding spectral sequences for the G-sheaf $f_*(A)$ over Y, and we have*

$$\mathbf{H}^q(G, A) \cong H^q(G, f_*(A)) \tag{5.2.11}$$

The first assertion follows immediately from the form of the initial term of the spectral sequence E defining (5.2.10). The other assertions follow from the explicit definition of the terms to be compared, if we note that for every injective resolution $\mathbf{C}(A)$ from A to $\mathbf{C}^{X(G)}$, the complex $f_*(\mathbf{C}(A))$ is a *resolution* of $f_*(A)$ (that is the significance of the assumption $R^q f_*(A) = 0$ for $q > 0$!) which is injective by virtue of Proposition 5.1.3. Note that we have implicitly defined the $R^q f^*(A)$ as right derived functors of f_* considered as functors from $\mathbf{C}^{X(G)}$ to $\mathbf{C}^Y$; but by virtue of the calculation of these derived functors by 3.7.2, recalling that an injective object of $\mathbf{C}^{X(G)}$ is a flabby sheaf, we see that we find the same result as if we consider f_* to be a functor from $\mathbf{C}^X$ to $\mathbf{C}^Y$, i.e. *$R^q f_*(A)$ is the sheaf over Y associated to the presheaf formed from the $H^q(f^{-1}(U), A)$.*

The hypothesis of Proposition 2.5.2 should be regarded as a cohomological equivalent of the hypothesis that G is a "discontinuous group of operators" over X. Another important case is the one in which $\mathbf{H}^q(G, A) = 0$ for $q > 0$, a hypothesis that should be regarded as the cohomological equivalent of the usual condition that "G operates without fixed points":

5.2.4 Proposition. *Suppose that $\mathbf{H}^q(G, A) = 0$ for $q > 0$, then $H^n(X; G, A) \cong H^n(Y, A^G)$, therefore $H^*(Y, A^G)$ is the abutment of a cohomological spectral sequence whose initial term is $\mathrm{II}_2^{p,q}(A) = H^p(G, H^q(X, A))$.*

This is an immediate consequence of the first spectral sequence of Theorem 2.5.1. We thus obtain the classical form of the theory of spaces with operators [4]. In the next section, we will give conditions under which this proposition holds, G being a "discontinuous group of operators on X without fixed point". Other conditions for validity are related to the characteristic of the base field k for the sheaf A and the orders of the stabilizers G_y; the following is a particularly simple case (which, moreover, can easily be directly proved, and is doubtless well known):

Corollary. *Assume (in addition to $\mathbf{H}^q(G, A) = 0$ for $q > 0$) that G is a finite group of order m, that the space X is separated, and that multiplication by m is an automorphism on A (for example, A is a sheaf of vector spaces over a field whose characteristic does not divide m). Then we have $H^n(Y, A^G) \cong H^n(X, A)^G$, the two terms being isomorphic to $H^n(X; G, A)$.*

A final interesting special case is the following:

5.2.5 Proposition. *Assume that* $H^q(X, A) = 0$ *for* $q > 0$. *Then*

$$H^n(X; G, A) \cong H^n(G, \Gamma(A))$$

and consequently $H^*(G, \Gamma(A))$ *is the abutment of a spectral sequence whose initial term is* $\mathrm{I}_2^{p,q}(A) = H^p(Y, \mathbf{H}^q(G, A))$.

This proposition can be used to calculate the cohomology of certain groups, for example the cohomology with constant coefficients of various modular groups in one variable [10].

Remark 1. Assume that we have a G-sheaf $\mathbf{O}$, and we restrict ourselves to G-$\mathbf{O}$-modules, Γ_X^G thus being considered a functor from $\mathbf{C}^{\mathbf{O}(G)}$ to the category of $\Gamma(\mathbf{O})^G$-modules. Then its derived functors coincide with the preceding functors $H^p(X; G, A)$ as immediately follows from the fact that for $n > 0$, $H^n(X; G, A)$ vanishes on injectives of $\mathbf{C}^{\mathbf{O}(G)}$. This latter fact will be proved in Lemma 5.6.1, below (replacing B by A).

Remark 2. We can, in the two exact sequences (5.2.8) and (5.2.9), eliminate $H^*(X; G, A)$ so that we obtain relations between the cohomology of Y, of X, and of G. More generally, assume that we have two five-term exact sequences as in the present case (cf. Diagram below).

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & A' \arrow[d, "\alpha"'] \arrow[dr, "u"] & & & \\
0 \arrow[r] & A \arrow[r, "\alpha'"] \arrow[dr, "u'"'] & H^1 \arrow[r, "\beta'"] \arrow[d, "\beta"'] & B \arrow[r, "\gamma'"] & C \arrow[r, "\delta'"] & D \\
 & & B' \arrow[d, "\gamma"'] & & & \\
 & & C' \arrow[d, "\delta"'] & & & \\
 & & D' & & &
\end{tikzcd}$$

We have set $u = \beta'\alpha$ and $u' = \beta\alpha'$. We have $A \cap A' = \operatorname{Ker} u = \operatorname{Ker} u'$. We have natural homomorphisms

$$\beta\beta'^{-1} : \operatorname{Ker}\gamma' \longrightarrow \operatorname{Coker} u'$$

$$\beta'\beta^{-1} : \operatorname{Ker}\gamma \longrightarrow \operatorname{Coker} u$$

These homomorphisms give rise to the following two exact sequences (in which H^1 has disappeared):

$$
(5.2.12)\qquad \begin{cases} 0 \longrightarrow \operatorname{Ker} u \longrightarrow A' \longrightarrow \operatorname{Ker}\gamma' \longrightarrow \operatorname{Coker} u' \longrightarrow C' \longrightarrow D' \\ \qquad\quad \| \\ 0 \longrightarrow \operatorname{Ker} u' \longrightarrow A \longrightarrow \operatorname{Ker}\gamma \longrightarrow \operatorname{Coker} u \longrightarrow C \longrightarrow D \end{cases}
$$

We find, for example: let u be the natural homomorphism from $H^1(Y, A^G)$ to $H^1(X, A)^G$, and let u' be the natural homomorphism from $H^1(G, \Gamma(A))$ to $H^0(Y, \mathbf{H}^1(G, A))$; the kernel of u is isomorphic to the kernel of u', while the image of u is isomorphic to the kernel of the natural homomorphism of $\operatorname{Ker}(H^1(X, A)^G \longrightarrow H^2(G, \Gamma(A))) \longrightarrow \operatorname{Coker} u'$.

Remark 3. The conditions of Proposition 5.2.2 are satisfied in most cases that arise naturally. Its interest lies mainly in formula (5.2.11), which, when G is finite, can also be written

$$
(5.2.11') \qquad \mathbf{H}^q(G, A)(y) = \mathbf{H}^q(G, f_*(A)(y)) \qquad y \in Y
$$

and gives an explicit calculation for the sheaves $\mathbf{H}^q(G, A)$. Proposition 5.2.2 applies whenever G is finite and X is separated. Another important case is the one in which G is a finite group of automorphisms of an abstract algebraic variety X such that $Y = X/G$ is an algebraic variety (for example, G is the Galois group of a covering space X, ramified or not of a normal variety Y): the conditions of Proposition 5.2.2 are then satisfied if A is a *coherent algebraic G-sheaf*, or if A is the multiplicative sheaf $\mathbf{O}_X^*$ of germs of invertible regular functions on X. Moreover, if X is unramified over Y, then we show that $\mathbf{H}^n(G, A) = 0$ for $n > 0$ when A is a coherent algebraic G-sheaf, which allows us to apply Proposition 5.2.3. In the same case, we can also show that $\mathbf{H}^1(G, \mathbf{O}_X^*) = 0$. We can also consider arithmetic varieties; the same results hold.

5.3 Case of a discontinuous group of homeomorphisms

For the sake of brevity, and by abuse of language, we will say that G is a *discontinuous group of homeomorphisms of* X if it satisfies the following condition:

(D) For every $x \in X$, the stabilizer G_x of x is finite, and there exists a neighborhood V_x of x such that for every $g \in G - G_x$, we have $g \cdot V_x \cap V_x = \emptyset$.

We can then obviously assume that V_x is open and also that $g \cdot V_x = V_x$ for $g \in G_x$ (replacing V_x, as required, by the intersection of the $g \cdot V_x$, $g \in G_x$). Condition (D) is satisfied, for example, if G is finite and X is separated, as we will soon see. If X is an irreducible algebraic variety, equipped with its Zariski topology, and G is a finite group of automorphisms of X, condition (D) *is not* satisfied (unless G operates trivially).

5.3.1 THEOREM. *We assume that condition (D) is satisfied. Let A be an abelian G-sheaf over X, $y \in Y$, and $x \in f^{-1}(y)$. We then have canonical isomorphisms*

$$\mathbf{H}^n(G, A)(y) \cong H^n(G_x, A(x)) \tag{5.3.1}$$

Taking V_x as in the statement of condition (D), with V_x open and stable under G_x, we can apply the corollary to Proposition 5.2.1 which reduces to the case in which $G = G_x$ and $X = V_x$, i.e. in which G is *finite*. We show that then $R^q f_*(A) = 0$ for $q > 0$. For every $y \in Y$, we have:

$$R^q f_*(A)(y) \cong \varinjlim H^q(f^{-1}(U), A)$$

with the limit taken over the open neighborhoods U of y. Then taking x and V_x as above, and restricting ourselves to $U \subseteq f(V_x)$, we see that $f^{-1}(U)$ is the finite union of pairwise disjoint open sets $g \cdot U_x$ ($g \in G/G_x$) where $U_x = V_x \cap f^{-1}(U)$, therefore $H^q(f^{-1}(U), A)$ is the direct sum of a finite number of groups isomorphic to $H^q(U_x, A)$. When U runs over a basic system of neighborhoods of y and U_x runs over a basic system of neighborhoods of x, then we get, for $R^q f_*(A)$, the direct sum of a finite number of groups isomorphic to $\varinjlim H^q(U_x, A)$, which is zero for $q > 0$ under Lemma 3.8.2.

We then have $R^q f_*(A) = 0$ for $q > 0$, therefore it follows from Proposition 5.2.2 that the left hand side of (5.3.1) can be identified as $\mathbf{H}^n(G, f_*(A))(y)$, which, since G is finite, is isomorphic to $H^n(G, f_*(A)(y))$ (as we noted in 4.4). Recalling that we can assume $X = V_x$, and therefore that $f^{-1}(y)$ is reduced to x, we then have $f_*(A)(y) = A(x)$, which completes the argument.

COROLLARY 1. *Assume that for every $x \in X$, the order n_x of the stabilizer G_x is such that multiplication by n_x is an automorphism of the group $A(x)$. Then $H^*(X; G, A) \cong H^*(Y, A^G)$ and consequently $H^*(Y, A^G)$ is the abutment of a spectral sequence whose initial term is $\mathrm{II}_2^{p,q} = H^p(G, H^q(X, A))$.*

In fact, Theorem 3.5.1 gives $\mathbf{H}^q(G, A) = 0$ for $q > 0$, so we can apply Proposition 5.2.3. The preceding corollary is particularly interesting in the case in which A is a sheaf of vector spaces over a field of characteristic p that does not divide any of the n_x (which is always true in characteristic 0). When there is no base field, we can use the following variant:

COROLLARY 2. *Assume that n is the least common multiple of the orders n_x of the G_x; let P be the set of prime divisors of n and $\mathbf{C}(P)$ be the category of abelian groups in which each element has finite order all of whose prime divisors lie in P. Then using the terminology introduced in 1.11, $H^*(X; G, A)$ is isomorphic mod $\mathbf{C}(P)$ to $H^*(Y, A^G)$, so that the latter is the abutment mod $\mathbf{C}(P)$ of a cohomological spectral sequence whose initial term is $H^p(G, H^q(X, A))$.*

To see this, reduce the spectral sequences of Theorem 5.2.1 mod $\mathbf{C}(P)$. We have, $\mathrm{I}_2^{p,q} = 0$ mod $\mathbf{C}(P)$ for $q > 0$, since under Theorem 5.3.1, $\mathbf{H}^q(G, A)$ is a sheaf annihilated by n, therefore $H^p(X, \mathbf{H}^q(G, A))$ is annihilated by n for every p and therefore is zero mod $\mathbf{C}(P)$. Corollary 2 follows immediately.

If the n_x are all reduced to 1, i.e. if every $g \neq e$ in G is fixed-point free (we then say simply that G *operates without fixed points*), we find $\mathbf{H}^n(G, A) = 0$ for $n > 0$. Thus $H^*(X; G, A) \cong H^*(Y, A^G)$, which is also evident a priori because of the isomorphism indicated at the end of 5.1 between the category of sheaves over Y and the category of G-sheaves over X, when G is a discrete group of fixed-point-free homeomorphisms. Then in this case, we find the classical statement:

Corollary 3. *If G operates without fixed points, then $H^*(Y, A^G)$ is the abutment of a cohomological spectral sequence whose initial term is $H^p(G, H^q(X, A))$.*

Remark. By definition, the second spectral sequence of Theorem 5.2.1 is obtained by taking a resolution $C(A)$ of A by Γ_X-acyclic sheaves, and by taking the second spectral sequence of the functor Γ^G with respect to the complex $\Gamma_X(C(A))$. In general, the first spectral sequence of Γ^G with respect to this complex clearly cannot be identified with the first spectral sequence of Theorem 5.2.1 (take an injective resolution of A!). However it works if we take for $C(A)$ the *canonical resolution* of A (cf. 3.3), as we see by an argument analogous to the one in Proposition 4.3.2. It also works if X and Y are paracompact and if we take for $C(A)$ a Cartan resolution of A, i.e. $C(A) = A \otimes_{\mathsf{Z}} C$, where C is a fundamental sheaf in the sense of Cartan-Leray, as we can show by rather different arguments (we will not have to make use of this fact). Consequently, the spectral sequence I is the spectral sequence in [10] whose initial term was not made explicit.

5.4 Transformation of the first spectral sequence

We will suppose that Condition (D) of the preceding section still holds (although this is not absolutely necessary). Let Y_0 be a closed subset of Y containing the supports of the $\mathbf{H}^n(G, A)$ for $n > 0$ (it suffices, for example, under Theorem 5.3.1, that Y_0 contain the set $\{y \in Y \mid G_y \neq e\}$). Let $X_0 = f^{-1}(Y_0)$, and let $V = \complement Y_0$ and $U = \complement X_0$. We consider the exact sequences

$$0 \longrightarrow A_U \longrightarrow A \longrightarrow A_{X_0} \longrightarrow 0$$

$$0 \longrightarrow (A^G)_V \longrightarrow A^G \longrightarrow (A^G)_{Y_0} \longrightarrow 0$$

as it is clear that $(A_U)^G = (A^G)_V$ we get a commutative diagram of cohomology sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^n(Y, (A^G)_V) \arrow[r] \arrow[d, "\cong"] & H^n(Y, A^G) \arrow[r] \arrow[d] & H^n(Y, (A^G)_{Y_0}) \arrow[r] \arrow[d] & H^{n+1}(Y, (A^G)_V) \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & H^n(X; G, A_U) \arrow[r] & H^n(X; G, A) \arrow[r] & H^n(X; G, A_{X_0}) \arrow[r] & H^{n+1}(X; G, A_U) \arrow[r] & \cdots \end{tikzcd} \tag{5.4.1}$$

It follows from the assumption on Y_0 that $\mathbf{H}^n(G, A_U) = 0$ for $n > 0$, since that is clearly the case for the open set V, but it is also true for Y_0, since according to Theorem 5.3.1, we have $\mathbf{H}^n(G, A_U)(y) = \mathbf{H}^n(G, A_U(y))$ for $y \in Y_0$. It follows from Proposition 5.2.3 that the homomorphism $H^n(Y, (A^G)_V) \longrightarrow H^n(X; G, A_U)$ is thus an isomorphism for all n. We also see just as easily that $H^n(X; G, A_{X_0})$ can be identified with $H^n(X_0; G, A)$; we also know that $H^n(Y, (A^G)_{Y_0}) \cong H^n(Y_0, A^G)$. We find, by chasing the preceding diagram, a sequence of homomorphisms

$$(5.4.2)\qquad \begin{aligned} \cdots \longrightarrow H^{n-1}(X; G, A) &\xrightarrow{\beta^{n-1}} \frac{H^{n-1}(X_0; G, A)}{\operatorname{Im} H^{n-1}(Y_0, A^G)} \xrightarrow{\partial} H^n(Y, A^G) \\ &\xrightarrow{\alpha^n} H^n(X; G, A) \xrightarrow{\beta^n} \frac{H^n(X_0; G, A)}{\operatorname{Im} H^n(Y_0, A^G)} \longrightarrow \cdots \end{aligned}$$

which is a complex (the composite of two successive homomorphisms is zero) and exact everywhere except possibly at the term $H^n(Y, A^G)$, the failure of exactness being that $\operatorname{Ker} \alpha^n / \operatorname{Im} \partial$ is canonically isomorphic to the kernel of the homomorphism

$$H^n(Y_0, A^G) \longrightarrow H^n(X_0; G, A)$$

. We thus obtain:

5.4.1 PROPOSITION. *The sequence of homomorphisms (5.4.2) defines over $H^n(Y, A^G)$ a sequence of length 3 whose successive quotients are isomorphic, respectively, to*

$$\operatorname{Coker} \beta^{n-1} = H^{n-1}(X_0; G, A)/(\operatorname{Im} H^{n-1}(X; G, A) + \operatorname{Im} H^{n-1}(Y_0, A^G))$$

$$\operatorname{Ker}(H^n(Y_0, A^G) \longrightarrow H^n(X_0; G, A))$$

and

$$\operatorname{Ker} \beta^n \subseteq H^n(X; G, A)$$

We can thus say that the use of the first spectral sequence gives us a way to reduce the determination of $H^*(Y, A^G)$ to a good understanding of $H^*(X; G, A)$, $H^*(X_0; G, A)$, $H^*(Y_0, A^G)$, and natural homomorphisms $H^*(X; G, A) \longrightarrow H^*(X_0; G, A) \longleftarrow H^*(Y_0, A^G)$. We can try to determine the first two groups by using the *second* spectral sequence while computing $H^*(Y_0, A^G)$ will often be possible if we can choose Y_0 sufficiently small. A particularly simple and important case is given by:

COROLLARY. *Assume that G operates trivially on X_0 (thus G is finite), that A is a sheaf of vector spaces over a field k, and that G acts trivially on $A|X_0$. Then the sequence (5.2.4) is exact, and $H^n(X_0; G, A)$ can be canonically identified with $\bigoplus_{p+q=n} H^p(G, k) \otimes H^q(X_0, A)$.*

In fact, the latter assertion is a particular case of the final statement of Theorem 4.4.1. Consequently, the canonical homomorphism from $H^n(Y_0, A^G) iso H^n(X_0, A)$ to $H^n(X_0; G, A)$ is injective, and therefore the sequence (5.2.4) is exact.

Remark. Assume that G is a finite group of prime order p. Then if we take for X_0 the set of fixed points of X under G, and $Y_0 = f(X_0)$, then Y_0 satisfies the condition stated at the beginning of this section. Then if a A is a sheaf of vector spaces coming from a sheaf over Y (i.e. such that G operates trivially on $A|X_0$), we have the conditions required for Corollary 1. For example, we can very easily recover the theorem of P. A. Smith relative to the case in which X is a homological sphere mod p of finite dimension (p being the order of G): then X_0 is also a homological sphere mod p. Moreover, this "natural" argument using general spectral sequences is, in fact, very similar to Borel's [2]; of course, the compactness hypothesis in [2] is unnecessary. It is even easier to find, as an application of the preceding corollary, that if a finite group G of prime order p operates on a space X of finite dimension which is acyclic mod p, then the set X_0 of fixed points and the quotient space X/G are also acyclic mod p. It follows that if X is acyclic with respect to a field k of coefficients (of arbitrary characteristic since the case of characteristic different from p is trivial), the same holds for X/G; since k is arbitrary, this claim remains valid if we replace k by $\mathbf{Z}$. Simple induction shows that these results for X/G remain valid if we assume merely that G is a finite solvable group. The computations in this section are also useful in studying Steenrod powers in sheaves and the cohomology of the symmetric powers of a space. We would like to mention that similar computations in an unpublished manuscript of R. Godement on Steenrod powers were the motivation for this section.

5.5 Computation of the $H^n(X; G, A)$ using covers

We will require the following:

5.5.1 Lemma. *Let $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ be three abelian categories, and assume that in the first two, every object is isomorphic to a subobject of an injective. Let $F_1 : \mathbf{C}_1 \longrightarrow \mathbf{C}_2$ and $F_2 : \mathbf{C}_2 \longrightarrow \mathbf{C}_3$ be covariant functors*[bb] *. We assume that F_1 takes injectives to F_2-acyclic objects. Let $\mathbf{K}$ be a covariant functor from $\mathbf{C}_1$ to the category $\mathbf{K}(\mathbf{C}_2)$ of complexes of positive degree in $\mathbf{C}_2$, satisfying the following two conditions: (i) $H^0\mathbf{K}(A) \cong F_1(A)$ (natural equivalence) and (ii) $\mathbf{K}(A)$ is acyclic in dimensions $n > 0$ when A is injective. Under these hypotheses, there exists in $\mathbf{C}_1$ a cohomological spectral functor abutting on the graded functor $(R^n(F_2F_1)(A))$, whose initial term is*

$$E_2^{p,q}(A) = \mathbf{R}^p F_2(R^q\mathbf{K}(A)) \tag{5.5.1}$$

We denote by H^0 the covariant functor $L \mapsto H^0(L)$ from $\mathbf{K}(\mathbf{C}_2) \longrightarrow \mathbf{C}_2$. Then we have $F_2F_1 = F_2(H^0\mathbf{K}) = F_2(H^0)\mathbf{K}$. Then if A is injective, $\mathbf{K}(A)$ is (F_2H^0)-acyclic, i.e. we have $\mathbf{R}^nF_2(\mathbf{K}(A)) = 0$ for $n > 0$. In fact, $\mathbf{K}(A)$ is a resolution of $F_1(A)$ as a consequence of

[bb] Translator's note: Grothendieck called the functors F and G and then went on to also use G to name the group used in the section title

(i) and (ii), therefore $\mathbf{R}^n F_2(\mathbf{K}(A)) = R^n F_2(F_1(A))$ and the final term is zero because of the assumption made on F_1. We can thus apply Theorem 2.4.1 to the composite functor $(F_2 H^0 \mathbf{K})$, which gives the spectral functor of the lemma.

5.5.2 LEMMA. *Let $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, F_1, F_2$ be as in the preceding lemma. There exists a cohomological spectral functor from $\mathbf{K}(\mathbf{C}_1)$ of complexes C of positive degree in $\mathbf{C}_1$, abutting on the graded functor $(\mathbf{R}^n(F_2F_1)(C))$, whose initial term is*

$$E_2^{p,q}(C) = \mathbf{R}^p F_2(R^q F_1(C)) \tag{5.5.2}$$

We indicate by a bar above a functor the functor obtained by extending it to categories of complexes of positive degree. We thereby get $\mathbf{R}^n(F_2F_1) \cong R^n(H^0\overline{F_2F_1}) \cong R^n((H^0\overline{F_2})\overline{F_1})$. The functor $\overline{F_1} : \mathbf{K}(\mathbf{C}_1) \longrightarrow \mathbf{K}(\mathbf{C}_2)$ takes injectives to $(H^0\overline{F_2})$-acyclic objects; in other words, if C is an injective object of $\mathbf{K}(\mathbf{C}_1)$, then $\mathbf{R}^n F_2(\overline{F_1}(C)) = 0$ for $n > 0$. In fact, in accordance with the characterization of injectives in $\mathbf{K}(\mathbf{C}_1)$ seen in 2.4, C is acyclic in dimensions > 0 and "decomposes"; moreover, the C^i and $Z^i(C)$ are injective, whence it follows first that $\overline{F_1}(C)$ is a resolution of $F_1(Z^0(C))$, from which we get $\mathbf{R}^n F_2(\overline{F_1}(C)) \cong R^n F_2(F_1(Z^0(C)))$, and since $Z^0(C)$ is injective, the second term is 0, from the assumption on F_1. Thus, we can apply Theorem 2.4 to the composite functor $(H^0\overline{F_2})\overline{F_1}$, which gives the spectral sequence in the statement, given that, by definition, $R^q\overline{F_1} = \overline{R^q}F_1$.

COROLLARY. *Under the preceding conditions, let C be a resolution of an object $A \in \mathbf{C}_1$. There is a cohomological spectral sequence, abutting on the graded object $(R^n(F_2F_1)(A))$ whose initial term is given in (5.5.2).*

In fact, what we have here is $\mathbf{R}^n(F_2F_1)(C) = R^n(F_2F_1)(A)$.

We return to the consideration of the space X with a group G of operators. Let $\mathbf{U} = (U_i)_{i\in I}$ be a cover of X, with G operating on the set I of indices in such a way that we have $g \cdot U_i = U_{g\cdot i}$ for every $i \in I$: we will say that $\mathbf{U}$ is a *G-cover*. If P is an abelian presheaf over X on which G operates, then the complex C of $\mathbf{U}, P$ of the cochains of $\mathbf{U}$ with coefficients in P can clearly be considered to be a complex of G-groups. We set

$$H^n(\mathbf{U};G,P) = \mathbf{R}^n\Gamma^G(C(\mathbf{U},P)) \tag{5.5.3}$$

in which the second term can described more explicitly as the group $H^n(C(G,C(\mathbf{U},P)))$ with $C(G,C(\mathbf{U},P))$ being the bicomplex formed from the cochains of G with values in the complex $C(\mathbf{U},P)$; this bicomplex can be denoted as $C(\mathbf{U};G,P)$:

$$H^n(\mathbf{U};G,P) = H^n(C(\mathbf{U};G,P)) \tag{5.5.3$'$}$$

We first assume that the G-cover $\mathbf{U}$ is *open*. We will apply Lemma 5.5.1, taking the categories $\mathbf{C}^{X(G)}$, $\mathbf{C}^G$ (the category of G-modules), and $\mathbf{C}$ (the category of abelian groups) and the functors Γ_X and Γ^G, and setting $\mathbf{K}(A) = C(\mathbf{U}, A)$. Γ_X takes injectives to Γ^G-acyclic objects (corollary to Proposition 5.1.3), and conditions (i) and (ii) of the lemma are satisfied, (i) because $\mathbf{U}$ is open and (ii) from the second paragraph of 3.8. It remains to make $R^q\mathbf{K}(A)$ explicit. The computation is immediate (and, moreover, was already carried out in 3.8): we find $R^q\mathbf{K}(A) \cong C(\mathbf{U}, H^q(A))$ where $H^q(A)$ denotes the presheaf $H^q(A)(V) = H^q(V, A)$. We have thus proved:

5.5.3 THEOREM. *Let X be a space equipped with a group G of operators. Let $\mathbf{U} = (U_i)$ be an open G-cover of X. Then on the category $\mathbf{C}^{X(G)}$ of abelian G-sheaves on X there exists a cohomological spectral functor abutting on the graded functor $(H^n(X; G, A))$, whose initial term is*

$$E_2^{p,q}(A) = H^p(\mathbf{U}; G, H^q(A)) \tag{5.5.4}$$

(where $H^q(A)$ is the presheaf $H^q(A)(V) = H^q(V, A)$ on X).

As usual we infer edge homomorphisms from this

$$H^n(\mathbf{U}; G, A) \longrightarrow H^n(X; G, A) \tag{5.5.5}$$

and a five term exact sequence, showing among other things that *the preceding homomorphism is injective when $n = 1$.* Moreover,

COROLLARY 1. *If the $U_{i_0,i_1,\ldots,i_p}$ are all A-acyclic, then the homomorphisms (5.5.5) are isomorphisms.*

COROLLARY 2. *The preceding theorem is still valid if $\mathbf{U}$ is a closed G-cover provided we have one of the following cases: (a) $\mathbf{U}$ is locally finite and X is paracompact; or (b) $\mathbf{U}$ is finite.*

In case (a), following remark 3.8.2, conditions (i) and (ii) of Lemma 5.5.1 still hold (for (i), it is moreover trivial without the assumption of paracompactness, since U is locally finite), $R^q\mathbf{K}(A) \cong C(\mathbf{U}, H^q(A))$ is still valid. In case (b), it is necessary to follow Godement's methods (which he developed for the one-element group), which we will sketch because of its potential application to abstract algebraic geometry. First, following an idea of P. Cartier, we introduce the sheaf complex $\mathbf{C}(\mathbf{U}, A)$, for which the complex of sections over an arbitrary open set V is, by definition, $C(\mathbf{U}|V, A|V)$ (where $|$ indicates, as usual, that we interpret the restriction as applying to the set following the $|$). We prove that this is a *resolution* of A (see Godement, [9] for details). We will apply to it the corollary to Lemma 5.5.2 (the letters in the statement of this lemma having the same meaning as above). We need prove only that $R^qF_1(C) = H^q(X, \mathbf{C}(\mathbf{U}, A))$ is isomorphic (naturally equivalent) to $C(\mathbf{U}, H^q(A))$. This presents no difficulty, since $\mathbf{U}$ is *finite* and we are using Theorem 3.5.1.

We will say that a *G-cover* $\mathbf{U} = (U_i)_{i\in I}$ is *"without fixed point"* if G operates on I without fixed points, i.e. if $g \neq e$ implies $g \cdot i \neq i$ for any $i \in I$. Then if P is an arbitrary G-presheaf, for any n we have

$$C^n(\mathbf{U}, P) = \prod_{(i_0,\dots,i_n)} \prod_{g\in G} P(g \cdot U_{i_0,i_1,\dots,i_n}) \tag{5.5.6}$$

with the first product taken over a complete set of representatives $(i_0, i_1, \dots, i_n)$ of I^{n+1}/G. In the second product, all the terms are canonically isomorphic to $P(U_{i_0,i_1,\dots,i_n})$ such that this product can be identified with the group of functions from G to the fixed group $P(U_{i_0,\dots,i_n})$, with G acting by left translation. It is well known that such a G-module is Γ^G-acyclic, thus the same is true for $C^n(\mathbf{U}, P)$. We will keep in mind, however, that formula (5.5.6) is valid only if we assume $C(\mathbf{U}, P)$ denotes the complex of *arbitrary* cochains (not necessary alternating) of $\mathbf{U}$ with coefficients in P (while earlier considerations were valid for either arbitrary or alternating cochains). Using the first hyperhomology spectral sequence of the functor Γ^G with respect to the complex $C(\mathbf{U}, P)$, we get:

5.5.4 PROPOSITION. *If U is a G-cover without fixed points, then for every G-presheaf P we have:*

$$H^n(\mathbf{U}; G, P) \cong H^n(C(\mathbf{U}, P)^G) \tag{5.5.7}$$

in which $C(\mathbf{U}, P)$ denotes the complex (with operators) of arbitrary cochains (not necessarily alternating) of $\mathbf{U}$ with coefficients in P.

Combining this with Corollary 1 of Theorem 5.5.3, we get:

COROLLARY. *Let $\mathbf{U}$ be an G-cover without fixed points. Assume that $\mathbf{U}$ is an open cover or that $\mathbf{U}$ is a closed cover and, in addition, that X is paracompact or that $\mathbf{U}$ is finite. If A is a G-sheaf such that $H^n(U_{i_0,i_1,\dots,i_p}, A) = 0$ for all $(i_0, i_1 \dots, i_p) \in I^{p+1}$ and every $n > 0$, then we have canonical isomorphisms:*

$$H^n(X; G, A) \cong H^n(C(\mathbf{U}, A)^G)$$

We are going to make explicit the complex $C(\mathbf{U}, P)^G$ when $\mathbf{U}$ is a G-cover without fixed points. Then the set of indices of $\mathbf{U}$ is isomorphic (as a set on which G operates) to a product $G \times I$ in which G operates by left translation on the first factor: $g' \cdot (g, i) = (g'g, i)$; we will identify i with (e, i). Let $(f_{g_0 i_0, g_1 i_1, \dots, g_n i_n})$ be an invariant n-cochain with coefficients in P. We set

$$F_{i_0, g_1 i_1, \dots, g_n i_n} = f_{e \cdot i_0, g_1 i_1, \dots, g_n i_n} \tag{5.5.9}$$

F is a function which depends the on $n+1$ arguments $i_0, i_1, \dots, i_n \in I$ and the n arguments $g_1, \dots, g_n \in G$, with values in the $P(U_{i_0} \cap g_1 U_{i_1} \cap \dots \cap g_n U_{i_n})$ and the invariant cochain given is entirely determined by the knowledge of this *"inhomogeneous cochain"* F according to

$$f_{g_0 i_0, g_1 i_1, \dots, g_n i_n} = g_0 \dot{F}_{i_0, g_0{}^{-1} g_1 i_1, \dots g_0{}^{-1} g_n i_n} \tag{5.5.10}$$

This formula defines, when we begin with an inhomogeneous cochain F, an invariant cochain f and the inhomogeneous cochain associated with f is nothing other than F. What remains is to express the differential operator on $C(\mathbf{U}, P)^G$ directly in terms of inhomogeneous cochains. We find immediately:

$$\begin{aligned}(\partial F)_{i_0, g_1 i_1, \dots, g_{n+1} i_{n+1}} &= g_1 . F_{i_1, g_1{}^{-1} g_2 i_2, \dots, g_1{}^{-1} g_{n+1} i_{n+1}} \\ &+ \sum_{1 \le \alpha \le n+1} (-1)^\alpha F_{i_0, g_1 i_1, \dots, \widehat{g_\alpha i_\alpha}, \dots, g_{n+1} i_{n+1}}\end{aligned} \tag{5.5.11}$$

where the sign ˆ signifies the omission of the term it is placed over. An especially interesting case is the one in which I is reduced to a single element, i.e. where we begin with a subset D of X such that $\bigcup_{g \in G} g \cdot D = X$ and consider the cover $(g \cdot D)_{g \in G}$. Then the inhomogeneous n-cochains consist of systems $(F_{g_1, \dots, g_n})$ with

$$F_{g_1, \dots, g_n} \in P(D \cap g_1 \cdot D \cap \cdots \cap g_n \cdot D)$$

and formula (5.5.11) becomes

$$\begin{aligned}(\partial F)_{g_1, \dots, g_{n+1}} &= g_1 \cdot F_{g_1{}^{-1} g_2, \dots, g_1{}^{-1} g_{n+1}} \\ &+ \sum_{1 \le \alpha \le n+1} (-1)^\alpha F_{g_1, \dots, \widehat{g_\alpha}, \dots, g_{n+1}}\end{aligned} \tag{5.5.11$'$}$$

Then let $L = \{g \in G \mid g \cdot D \cap D \neq \emptyset\}$ (this will be a finite set in "reasonable" cases). The preceding formula shows that the complex $C(\mathbf{U}, P)^G$ giving the groups $H^n(\mathbf{U}; G, P)$ can be constructed once we know: the composition law $(g, g') \mapsto g^{-1} g'$ in L (insofar as it is defined); the intersections $D \cap g \cdot D$, for $g \in L$; and of course the groups $P(D \cap g_1 \cdot D \cap \cdots \cap g_n \cdot D)$, for $g_i \in L$; the restriction functions; and the way the $g_i \in L$ operate on them.

Remark 1a. Under the preceding conditions, assume for example that P is the constant sheaf defined by a commutative ring k and that the $D \cap g_1 \cdot D \cap \cdots \cap g_n \cdot D$ are acyclic for k, and connected. Then the values $F_{g_1, \dots, g_n}$ are simply elements of k (defined for $D \cap g_1 \cdot D \cap \cdots \cap g_n \cdot D \neq \emptyset$), and the complex of these cochains gives the cohomology $H^*(X; G, k)$. This gives an explicit method of computation if we assume that the set L above is finite (which implies that the complex $C(\mathbf{U}, k)^G$ is free of finite rank in all dimensions). It seems that this should allow, for example, computation of the cohomology of modular groups in several variables (in this case $H^*(X; G, \mathbf{Z}) \cong H^*(G, \mathbf{Z})$, since X is a contractible open set in a Euclidean space), when we know a sufficiently simple fundamental domain D (without having to look at what happens at multiple points as in method [10], which does not seems feasible for a fixed-point set that is too complicated).

REMARK 1B. Take $D = X$; thus $\mathbf{U} = (g \cdot X)_{g \in G}$. We note that the spectral sequence of Theorem 5.5.3 is just the second spectral sequence of Theorem 5.2.1, whose initial term is $H^p(G, H^q(X, A))$. We also note that in every case there is a canonical natural transformation from the spectral sequence of Theorem 5.5.3 to the second spectral sequence of Theorem 5.5.1, as we can see by applying Lemma 5.5.1, respectively 5.5.2 to this end.

To complete this section, we will give some hints for the Čech computation of $H^n(X; G, A)$, generalizing the analysis of 3.8. Let $\mathbf{U}$ and $\mathbf{V}$ two G-covers, with $\mathbf{V}$ finer than $\mathbf{U}$. We will define canonical homomorphisms

$$H^n(\mathbf{U}; G, P) \longrightarrow H^n(\mathbf{V}; G, P) \tag{5.5.12}$$

defined for every abelian G-sheaf P and natural in P. First, setting $\mathbf{U} = (U_i)_{i \in I}$ and $\mathbf{V} = (V_j)_{j \in J}$, we assume that there exists a function $\phi : J \longrightarrow I$ such that $V_j \subseteq U_{\phi(j)}$ for all $j \in J$ which *commutes with the action of* G. From this we classically induce a homomorphism $\overline{\phi} : C(\mathbf{U}, P) \longrightarrow C(\mathbf{V}, P)$, which commutes with the action of G and thus, in accordance with (5.5.3), gives the desired homomorphism (5.5.12), which is *independent of the choice of* ϕ. If ϕ' is another function with the same properties, we classically construct a well-defined homotopy s from $\overline{\phi} \longrightarrow \overline{\phi'}$ such that $\overline{\phi} - \overline{\phi'} = \partial s + s\partial$, and for obvious reasons of "transport of structure", s commutes with the action of G, whence it follows that $\overline{\phi}$ and $\overline{\phi'}$ define the same homomorphism (5.5.12). If we no longer assume that we can find a ϕ as above, we consider the set $I' = G \times I$ on which we make G operate as $g'(g, i) = (g'g, i)$, and the G-cover $\mathbf{U}'$ without fixed points $(U'_{(g,i)})_{(g,i) \in G \times I}$ defined by $U'_{(g,i)} = U_{g \cdot i} = gU_i$. The function $\psi(g, i) = g \cdot i$ has the properties required above, and thus defines a homomorphism:

$$H^n(\mathbf{U}; G, P) \longrightarrow H^n(\mathbf{U}'; G, P)$$

Thus it suffices to define a canonical homomorphism $H^n(\mathbf{U}'; G, P) \longrightarrow H^n(\mathbf{V}; G, P)$, so we are now in the situation as above, since $\mathbf{U}'$ is a G-cover *without fixed points* that is refined by $\mathbf{V}$. The definition of homomorphisms (5.5.12) is thus complete; moreover, these homomorphisms have obvious transitivity properties. From this it follows first that if $\mathbf{U}$ and $\mathbf{V}$ are two equivalent G-covers, then the homomorphism (5.5.12) is an isomorphism so that $H^n(\mathbf{U}; G, P)$ depends only on the class of the G-cover $\mathbf{U}$ (for the equivalence relation defined by the preorder relation: $\mathbf{V}$ is finer than $\mathbf{U}$). We then set

$$\check{H}^n(X; G, P) = \varinjlim H^n(\mathbf{U}; G, P) \tag{5.5.13}$$

with the inductive limit taken over the ordered set of classes of open G-covers of X. The homomorphisms (5.5.3) define natural transformations

$$\check{H}^n(X; G, A) \longrightarrow H^n(X; G, A) \tag{5.5.14}$$

for $A \in \mathbf{C}^{X(G)}$. We propose to indicate the conditions under which these are isomorphisms. By passing to the inductive limit of the spectral sequences associated with the various $\mathbf{U}$

(Theorem 5.5.3), we can show that $H^*(X; G, A)$ is the abutment of a cohomological spectral functor, whose initial term is

$$E_2^{p,q}(A) = H^p(X; G, H^q(A)) \tag{5.5.15}$$

The homomorphisms of (5.5.14) are exactly the edge homomorphisms of this spectral sequence. Now we can prove:

5.5.5 Lemma. *Assume that Y is paracompact and X is separated. Let G be a discontinuous group of homeomorphisms on X (cf. 5.5.3). If P is a presheaf of abelian G-groups whose associated sheaf is 0, then $H^n(X; G, P) = 0$ for every $n > 0$.*

In computing $\check{H}^n(X; G, P)$ in accordance with (5.5.13), we can restrict ourselves to G-covers $\mathbf{U}$ without fixed points; for these, Proposition 5.5.4 applies so that it suffices to prove the following: if $f^n \in C^n(\mathbf{U}, P)^G$, where $\mathbf{U}$ is an open G-cover without fixed points, then there exists a finer open G-cover V without fixed points and a function $\phi : J \longrightarrow I$ between the sets of indices satisfying the above conditions and such that $\overline{\phi}(f^n) = 0$. We start by showing that there exists arbitrarily fine G-covers of the type $(g, U_i)_{(g,i)\in G\times I}$ in which (U_i) is a family of open sets of X such that if G_i is the stabilizer of U_i, then G_i is finite, and $g \notin G_i$ implies $gU_i \cap U_i = \emptyset$. We will thus assume that $\mathbf{U}$ is of the above type. We consider the inhomogeneous cochain $(F_{i_0,g_1i_1,\ldots,g_ni_n})$ which corresponds to f^n. Consider I as the nerve of the cover $(f(U_i)) = (U'_i)$ of Y and for every simplex $s = (i_0, i_1, \cdots, i_n)$ of I let $U'_s = U'_{i_0,i_1,\ldots,i_n}$[cc]. For every n-simplex s of I, and every $y \in U'_s$, let W^s_y be an open neighborhood of y contained in U'_s, such that for every system $\mathbf{g} = (g_1, \ldots, g_n)$ for which $U_{s,\mathbf{g}} = U_{i_0} \cap g_1U_{i_1} \cap \cdots \cap g_nU_{i_n}$ has a projection on Y which contains y; and the restriction of $(F_{i_0,g_1i_1,\ldots,g_ni_n})$ to $U_{s,\mathbf{g}} \cap f^{-1}(W^s_y)$ is 0. Such a W^s_y exists because of the assumption on P, because there are only a finite number of systems $\mathbf{g}$ to consider, and because, when W runs over a basic system of neighborhoods of y, the $U_{s,\mathbf{g}}\ capf^{-1}(W)$, run over a basic system of neighborhoods of the finite set of those $x \in U_{s,\mathbf{g}}$ that project on y. Since Y is paracompact, it follows from the well-known lemma in Čech theory that we can find a cover $\mathbf{V}' = (V'_j)_{j\in J}$ of Y, and a function $\phi' : J \longrightarrow I$ such that $V'_j \subseteq U'_{\phi'(j)}$ for every $j \in J$, and such that for every n-simplex t of J, V'_t is contained in at least one of the sets $W^{\phi'(t)}_y$. Then let $V_j = U_{\phi'(j} \cap f^{-1}(V'_j)$, let $V = (g \cdot V_j)_{(g,j)\in G\times J}$, and consider the function $\phi : G \times J \longrightarrow G \times I$ defined by ϕ'. We can immediately see that $(\mathbf{V}, \phi)$ satisfies the desired conditions, notably $\phi(F) = 0$, whence $\overline{\phi}(f^n) = 0$.

The sheaf associated to the presheaf $H^q(A)$ for $q > 0$ is 0 (Lemma 3.8.2). Lemma 5.5.5 and the spectral sequence whose initial term is (5.5.15) then show:

[cc] Translator's note: This sentence is so incoherent in the original that we have had to guess what was intended.

5.5.6 Theorem. *If G is a discrete group of homeomorphisms of X and if $Y = X/G$ is paracompact, then the homomorphisms of (5.5.14):*

$$\check{H}^n(X;G,A) \longrightarrow H^n(X;G,A)$$

are isomorphisms for every abelian G-sheaf A.

Remark 2. From the spectral sequence whose initial term is (5.5.15), we derive, without any assumptions on X, G, or A that

$$\check{H}^1(X;G,A) \cong H^1(X;G,A)$$

(and that $\check{H}^2(X;G,A) \longrightarrow H^2(X;G,A)$ is injective since we can show that $E_2^{0,n} = 0$ for all $n > 0$). $\check{H}^1(X;G,A)$ can be interpreted geometrically, as we can easily see, as the set of classes of G-A-fiber spaces over X (i.e. fiber spaces over X "with structure sheaf A", [11], on which G-operates compatibly). Moreover, $\check{H}^1(X;G,A)$ is also defined, and allows the same geometric interpretation, if A is a G-sheaf of not necessarily commutative groups. The development in [11], in particular in Chapter 5, can be carried over nearly word for word to the more general context of G-fiber spaces. A good suggestion would be to develop non-commutative homological algebra, in the context of functors and categories, in a direction that includes both this theory of fiber spaces and the algebraic mechanism for extensions of Lie groups and other groups, as developed in papers by G. Hochschild, [12, 13] and A. Shapiro [19].

5.6 The groups $\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B)$

In this section, $\mathbf{O}$ denotes a fixed sheaf of G-rings, and we consider the abelian category $\mathbf{C}^{\mathbf{O}(G)}$ of G-$\mathbf{O}$-modules (cf. 5.1). The group of $G-\mathbf{O}$-homomorphisms from one $G-\mathbf{O}$-module A to another $G-\mathbf{O}$-module B will be denoted $\mathrm{Hom}_{\mathbf{O},G}(A,B)$, or $\mathrm{Hom}_{\mathbf{O},G}(X;A,B)$ if we want make the space X over which we are considering A and B explicit (which also gives a meaning to the symbol $\mathrm{Hom}_{\mathbf{O},G}(U;A,B)$ if U is a G-invariant subset of X). Noting that the sheaf $\mathbf{Hom_O}(A,B)$ is a G-sheaf and thus the group $\mathrm{Hom}_{\mathbf{O}}(A,B) = \Gamma_X \mathbf{Hom_O}(A,B)$ is a G-module, we have the formulas

$$\mathrm{Hom}_{\mathbf{O},G}(A,B) \cong \Gamma_X^G \mathbf{Hom_O}(A,B) \cong \Gamma^G \mathrm{Hom}_{\mathbf{O}}(A,B) \tag{5.6.1}$$

We set

$$\mathrm{Hom}_{\mathbf{O},G}(A,B) = (\mathbf{Hom_O}(A,B))^G \tag{5.6.2}$$

therefore $\mathbf{Hom}_{\mathbf{O},G}(A,B)$ is the sheaf over $Y = X/G$ whose sections over the open set $U \subseteq Y$ form the group $\mathrm{Hom}_{\mathbf{O},G}(f^{-1}(U);A,B)$ of the $G-\mathbf{O}$-homomorphisms from $A|f^{-1}(U)$ to

$B|f^{-1}(U)$. We set

$$\begin{aligned} \mathbf{h}_{\mathbf{O},A}(B) &= \mathbf{Hom}_{\mathbf{O}}(A,B), \qquad h_{\mathbf{O},A}(B) = \mathrm{Hom}_{\mathbf{O}}(A,B) \\ \mathbf{h}_{\mathbf{O},G,A}(B) &= \mathbf{Hom}_{\mathbf{O},G}(A,B), \qquad h_{\mathbf{O},G,A}(B) = \mathrm{Hom}_{\mathbf{O},G}(A,B) \end{aligned} \tag{5.6.3}$$

thereby defining, for fixed A, four left exact functors, the last two being related to the first two (already encountered in Chapter 4) by the natural equivalences

$$\mathbf{h}_{\mathbf{O},G,A} \cong f_*^G \mathbf{h}_{\mathbf{O},A} \tag{5.6.4}$$

$$h_{\mathbf{O},G,A} \cong \Gamma_Y \mathbf{h}_{\mathbf{O},G,A} \cong \Gamma^G \mathbf{h}_{\mathbf{O},A} \cong \Gamma_X^G \mathbf{h}_{\mathbf{O},A} \tag{5.6.5}$$

Let us set

$$\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B) = R^n h_{\mathbf{O},G,A}(B) \tag{5.6.6}$$

$$\mathbf{Ext}^n_{\mathbf{O},G}(X;A,B) = R^n \mathbf{h}_{\mathbf{O},G,A}(B) \tag{5.6.7}$$

Thus for two $G-\mathbf{O}$-modules A, B, the $\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B)$ are abelian groups, forming a universal cohomological functor in B and reducing to $\mathrm{Hom}_{\mathbf{O},G}(X;A,B)$ is dimension 0, while the $\mathbf{Ext}^n_{\mathbf{O},G}(A,B)$ are abelian sheaves over Y, forming a universal cohomological functor in B and reducing to $\mathbf{Hom}_{\mathbf{O},G}(A,B)$ in dimension 0. Moreover, the $\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B)$ are just the general Ext groups in the category $\mathbf{C}^{\mathbf{O}(G)}$. (Of course, our definitions are justified by Proposition 5.1.1.) Since $\mathrm{Hom}_{\mathbf{O},G}(A,B)$ and therefore $\mathbf{Hom}_{\mathbf{O},G}(A,B)$ are exact contravariant functors in A whenever B is injective in $\mathbf{C}^(\mathbf{O}(G))$, it follows (cf. 2.3) that the $\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B)$, respectively $\mathbf{Ext}^n_{\mathbf{O},G}(A,B)$, also form contravariant cohomological functors in A (for fixed B). Finally, we note that we can easily show, as in 4.2 and 5.2 (which are special cases), that we have

$$\mathbf{Ext}^n_{\mathbf{O},G}(A,B)|U = \mathbf{Ext}^n_{\mathbf{O},G}(A|f\inf(U), B|f^{-1}(U)) \tag{5.6.8}$$

(*local nature of the* **Ext** with respect to the space Y). From this we derive a statement analogous to the statement in the corollary to Proposition 5.2.1, which in the case that G is a *discontinuous group* of homomorphisms (cf. 5.3), reduces to the case in which G is finite. Then we show, again using the reasoning in 4.1, that the natural transformations

$$\mathbf{Hom}_{\mathbf{O},G}(A,B)(y) \longrightarrow \mathrm{Hom}_{U_x}(A(x),B(x)) \qquad (f(x)=y) \tag{5.6.9}$$

(in which U_x is the ring generated by $\mathbf{O}(x)$ and $\mathbf{Z}(G)$ as in 5.1) are bijective when $\mathbf{O}$ is a *coherent* sheaf of *left Noetherian* rings, and A is coherent as an $\mathbf{O}$-module; and given these conditions on X, G, and $\mathbf{O}$, the U_x-module $B(x)$ is injective whenever B is an injective object of $\mathbf{C}^{\mathbf{O}(G)}$. (In the proof, we will replace the free $\mathbf{O}$-modules from 4.1 by $G-\mathbf{O}$-modules of the type $L(U)$ introduced in the proof of Proposition 5.1.1.) We conclude immediately *that the natural transformations*

$$\mathbf{Ext}^n_{\mathbf{O},G}(A,B)(y) \longrightarrow Ext^n_{U_x}(A(x),B(x)) \tag{5.6.10}$$

deduced from the homomorphisms (5.6.9) are isomorphisms if we assume that the group G of homeomorphisms is discontinuous in sense of 5.3, that $\mathbf{O}$ is a coherent sheaf of left Noetherian rings, and that A is a $G-\mathbf{O}$-module, coherent as an $\mathbf{O}$-module. This statement contains both Theorem 4.2.2 and Theorem 5.3.1 (the sheaf Z of constant rings being Noetherian and coherent!), and in the most important cases clarifies the structure of $\mathbf{Ext}^n_{\mathbf{O},G}(A,B)$.

By applying Theorem 2.4.1 we can obtain spectral sequences from the Formulas (5.6.4) and (5.6.5). We are interested only in the spectral sequences from (5.6.5), which abut on $\mathrm{Ext}^*_{\mathbf{O},G}(X;A,B)$. We still have to verify that in each of these three formulas giving $h_{\mathbf{O},G,A}$ as a composite functor, the usual acyclicity is satisfied; this results from

5.6.1 Lemma. *Assume that B is an injective object of $\mathbf{C}^{\mathbf{O}(G)}$. Then $\mathbf{Hom}_{\mathbf{O},G}(A,B)$ is a flabby sheaf over Y, $\mathrm{Hom}_{\mathbf{O}}(A,B)$ is a Γ^G-acyclic G-module, and $\mathbf{Hom}_{\mathbf{O}}(A,B)$ is a Γ^G_X-acyclic G-sheaf.*

From Proposition 5.1.2, this reduces to the case in which B is the product sheaf defined by a family $(B_x)_{x\in X}$ of injective U_x-modules B_x. It follows that the abelian G-sheaf $\mathbf{H} = \mathrm{Hom}_{\mathbf{O}}(A,B)$ is the product sheaf defined by the family of G-modules $H_x = \mathrm{Hom}_{\mathbf{O}(x)}(A(x),B(x))$. Then $\mathbf{Hom}_{\mathbf{O},G}(A,B)$ is the product sheaf over Y defined by the family of groups $H_y = \prod (H_x)^{Gx}$, the product taken over all the $x \in f^{-1}(y)$, and thus is flabby. We assume for the moment that the H_x are Γ^{Gx}-acyclic G_x-modules (this will be shown in the corollary below). Then the G-module $\overline{H}_x$ induced by the G_x-module H_x by contravariant extension of scalars is Γ^G-acyclic, as is well-known. Thus $\mathrm{Hom}_{\mathbf{O}}(A,B) = \Gamma_X\mathbf{H} = \prod_{x\in X/G} \overline{H}_x$ is Γ^G-acyclic. Finally, since $\mathbf{H}$ is flabby and thus Γ_X-acyclic, it follows from the second spectral sequence of Theorem 5.2.1 that $H^n(X;G,\mathbf{H}) = H^n(G,\Gamma_X\mathbf{H})$, which according to what we have seen vanishes for $n > 0$. This completes the proof of Lemma 5.6.1, provided we show the following special case:

Corollary. *Let O be a ring with unit on which a group G operates, and let U be the ring generated by O and $\mathsf{Z}(G)$ subject to the commutation relations $g\lambda g^{-1} = \lambda^g$ for $g \in G$ and $\lambda \in O$. Let A and B be two U-modules, with B injective. Then the G-module $Hom_{\mathbf{O}}(A,B)$ is Γ^G-acyclic. (But do not assume it is injective!)*

Let $B' = \mathrm{Hom}_{\mathsf{Z}}(\mathsf{Z}(G),B)$ be the group of functions $f : G \longrightarrow B$. We will make G and O operate on it by

$$(g'f)(g) = f(gg'), \qquad (\lambda f)(g) = \lambda^g f(g)$$

We immediately see that B' thus becomes a U-module and we define an injective U-homomorphism $\phi : B \longrightarrow B'$ by setting

$$\phi(b)(g) = g \cdot b$$

Thus B is embedded in the U-module B' and, since B is injective, it is a direct factor of B', thus $\mathrm{Hom}_O(A, B)$ is (as a G-module) a direct factor of $\mathrm{Hom}_O(A, B')$, and it is sufficient to show that the latter is Γ^G-acyclic. We set $H = \mathrm{Hom}_O(A, B)$ and we consider H to a G-module, making G operate on it by $u^g(a) = gu(g^{-1}a)$; we can immediately show that there is a canonical isomorphism

$$\mathrm{Hom}_O(A, B') \cong \mathrm{Hom}_O(A, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}(G), B)) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}(G), \mathrm{Hom}_O(A, B))$$

by imposing on the final term $H'' = \mathrm{Hom}_G(\mathbf{Z}(G), H)$ the G-module structure defined by $f^{g'}(g) = g'f(gg')$. For this, we write that, for every $u \in \mathrm{Hom}_O(A, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}(G), B))$, $u(a)(g) = g \cdot u_g(a)$; then the u_g are O-homomorphisms from A to B and u is identified with the family of u_g identified as an element of $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}(G), H)$. We also consider the G-module H' obtained by imposing on $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}(G), H)$ the G-module structure defined by $(g'f)(g) = f(gg')$. We obtain a $\mathbf{G}$-isomorphism ψ from H' to H'' by setting $\psi(f)(g) = g^f(g)$. It is well known that H' is Γ^G-acyclic for any G-module H; the same is therefore true for H''. Q.E.D.

Lemma 5.6.1 allows us to derive from the Formulas (5.6.5) three spectral sequences abutting on $\mathrm{Ext}_{\mathbf{O},G}(X; A, B)$. To make the initial terms of the first two explicit, it remains to specify the derived functors of $h_{\mathbf{O},A}$ and $\mathbf{h}_{\mathbf{O},A}$ considered as functors over on $\mathbf{C}^{\mathbf{O}(G)}$. When we consider them as functors on the category $\mathbf{C}^{\mathbf{O}}$ of $\mathbf{O}$-modules (without operators), their derived functors are, by definition (4.2), the functors $\mathrm{Ext}^n_{\mathbf{O}}(X; A, B)$ and the functors $\mathbf{Ext}^n_{\mathbf{O}}(A, B)$. This remains true when we consider them to be functors on $\mathbf{C}^{\mathbf{O}(G)}$ as immediately follows from the definitions and from

5.6.2 LEMMA. *If B is an injective $G - \mathbf{O}$-module, it is an injective $\mathbf{O}$-module.*

As in Lemma 5.6.1, this can be reduced to the case in which B is the product sheaf defined by a family $(B_x)_{x \in X}$ of injective U_x-modules B_x. It then suffices to prove that the B_x are also injective $\mathbf{O}(x)$-modules, which reduces to the case in which X is a single point x. In fact, the proof does not seem simpler in this case, and the proof we are going to give could in fact apply whenever we have an abelian category $\mathbf{C}$ (such as $\mathbf{C}^{\mathbf{O}}$) in which a group G "operates" as in footnote 9. This allows us to pass from $\mathbf{C}$ to the corresponding category $\mathbf{C}(G)$ (here $\mathbf{C}^{\mathbf{O}(G)}$ of "objects with operators". We assume that $\mathbf{C}$ satisfies AB 5) and has a generator: *then every injective object A of $\mathbf{C}(G)$ is also injective in $\mathbf{C}$* . To prove this, it is clearly sufficient to prove that every object A of $\mathbf{C}(G)$ is embedded in an object M of $\mathbf{C}(G)$ which is injective in $\mathbf{C}$. To do so, it is sufficient to examine the construction given in 1.10 for the embedding of A (considered as an object of $\mathbf{C}$) into an injective object

M of $\mathbf{C}$: we embed A in the module $M(A)$, which is a functor (not additive!) of A, since the homomorphism $A \longrightarrow M_1(A)$ is natural; we iterate the process transfinitely and pass to the inductive limit, which gives us an injective object $M(A)$, which is still *functorial* in A, and an injection $A \longrightarrow M(A)$ which is a *natural* homomorphism. This construction is well defined, once we have chosen a generator U of $\mathbf{C}$ and an appropriate cardinal number. Moreover, if we replace U, if necessary, by the direct sum of its images under G (cf. the proof of Proposition 5.1.1), we can assume that U is an object of $\mathbf{C}(G)$. We then see, for obvious reasons of "transport of structure", that if A is an object of $\mathbf{C}(G)$, G also operates on $M(A)$, which is then found to be in reality an object of $\mathbf{C}(G)$, and the injection $A \longrightarrow M(A)$ is compatible with the operations of G. This is the injection we seek from A to an object of $\mathbf{C}(G)$ which is injective in $\mathbf{C}$. This completes the proof; readers who are still doubtful can provide the details of the proof in the case $\mathbf{C}^{\mathbf{O}}$ of the lemma and, if they prefer, limit themselves to the "purely algebraic" case in which X is reduced to a single point.

We can now state the main result of this section, whose proof is contained in the preceding discussion.

5.6.3 THEOREM. *Let X be a space equipped with a group G of homeomorphisms and let $\mathbf{O}$ be a G-sheaf of rings on X. Let A be a $G-\mathbf{O}$-module. On the category $\mathbf{C}^{\mathbf{O}(G)}$ of $G-\mathbf{O}$-modules, we can find three cohomological spectral functors abutting on the graded functor $(\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B))$, whose initial terms are, respectively:*

$$\begin{aligned} \mathrm{I}_2^{p,q}(B) &= H^p(Y, \mathbf{Ext}^q_{\mathbf{O},G}(A,B)) \\ \mathrm{II}_2^{p,q}(B) &= H^p(G, \mathrm{Ext}^q_{\mathbf{O}}(X;A,B)) \\ \mathrm{III}_2^{p,q}(B) &= H^p(X; G, \mathbf{Ext}^q_{\mathbf{O}}(A,B)) \end{aligned} \tag{5.6.11}$$

If $\mathbf{O} = A$, the first two spectral sequences can be reduced to the spectral sequences of Theorem 5.2.1; the third spectral sequence is new. If G is the one element group, then spectral sequences I and III are identical, and also identical to the spectral sequence of Theorem 4.2.1, while spectral sequence II is trivial. If X is a one point space, spectral sequences II and III are identical, and moreover are non-trivial in general (and, it seems, useful), while spectral sequence I is trivial.[11]

From the preceding spectral sequences, we derive edge homomorphisms and five-term exact sequences, which we leave to the reader. We will explicitly state only a single degenerate special case:

[11] The non-trivial spectral sequence obtained abuts on $\mathrm{Ext}_U(A,B)$ and its initial term is $H^p(G, \mathrm{Ext}^q_O(A,B))$ (U being the ring generated by $\mathbf{Z}(G)$ and the G-ring O, with commutation relations $g\lambda g^{-1} = \lambda^g$, for $\lambda \in O$ and $g \in G$). We should point out that we can more easily obtain this spectral sequence directly, by using projective resolutions of A instead of injective resolutions of B. In fact, we can very easily see that if A is a free U-module, then A is a free O-module and $\mathrm{Hom}_O(A,B)$ is a Γ^G-acyclic G-module.

Corollary 1. *If the $G-\mathbf{O}$-module A is locally isomorphic, as an $\mathbf{O}$-module, to $\mathbf{O}^n$, then we have canonical isomorphisms:*

$$\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B) \cong H^n(X;G,\mathbf{Hom}_{\mathbf{O}}(A,B)) \tag{5.6.12}$$

This follows from spectral sequence III since then $\mathbf{Ext}^q_{\mathbf{O}}(A,B)=0$ for $q>0$ (Proposition 4.2.3). In particular, taking $A=\mathbf{O}$, we get:

Corollary 2. *For every $G-\mathbf{O}$-module B, we have natural equivalences:*

$$\mathrm{Ext}^n_{\mathbf{O},G}(X;\mathbf{O},B) \cong H^n(X;G,B) \tag{5.6.14}$$

Remark. We point out, in two special cases, two other spectral sequences abutting on $\mathrm{Ext}^n_{\mathbf{O},G}(X;A,B)$. Assume that $\mathbf{O}$ is the constant G-sheaf of rings defined by a ring O on which G operates, and that A is the constant $G-\mathbf{O}$-sheaf $\mathbf{M}$ defined by an O-module M on which G operates compatibly with its operations on O, namely $g\cdot(\lambda m)=g(\lambda)g(m)$. we introduce the ring U generated by O and $\mathbf{Z}(G)$ subject to the commutation relations $g\lambda g^{-1}=g(\lambda)$; M is thus a U-module. Consider the functor $h_{U,M}$ on the category of U-modules, defined by $h_{U,M}(N)=\mathrm{Hom}_U(M,N)$; we then have, for every $G-\mathbf{O}$-module B, $\mathrm{Hom}_{\mathbf{O},G}(\mathbf{M},B)=\mathrm{Hom}_U(M,\Gamma_X(B))$, i.e. we have the natural identity

$$h_{\mathbf{O},G,M}=h_{U,M}\Gamma_X \tag{5.6.5$'$}$$

In addition, we can easily verify that Γ_X transforms injective objects in $\mathbf{C}^{\mathbf{O}(G)}$ into injective U-modules, so that Theorem 2.4.1 gives a cohomological spectral functor IV on $\mathbf{C}^{\mathbf{O}(G)}$, abutting on the graded functor $(\mathrm{Ext}^n_{\mathbf{O},G}(X;\mathbf{M},B))$, whose initial terms is

$$\mathrm{IV}_2^{p,q}(B)=\mathrm{Ext}^p_U(M,H^q(X,B)) \tag{5.6.11$'$}$$

If, in addition, we assume that G operates trivially on M, then we also have $\mathrm{Hom}_{\mathbf{O},G}(\mathbf{M},B)\cong \mathrm{Hom}_U(M,\Gamma_X(B))\cong \mathrm{Hom}_{\mathbf{O}}(M,\Gamma_X^G(B))$, which gives the natural equivalence

$$h_{\mathbf{O},G,\mathbf{M}}\cong h_{O,M}\Gamma_X^G \tag{5.6.5 ter}$$

We can also show that Γ_X^G transforms injective objects of $\mathbf{C}^{\mathbf{O}(G)}$ into injective O-modules (by showing that if N is an injective U-module, then N^G is an injective O-module), and Theorem 2.4.1 gives a fifth cohomological spectral functor, with the same abutment as the preceding ones, whose initial term is

$$\mathrm{V}_2^{p,q}(B)=\mathrm{Ext}^p_{\mathbf{O}}(M,H^q(X;G,A)) \tag{5.6.11 ter}$$

If G is the one-element group, spectral sequences IV and V coincide with the spectral sequence of Theorem 4.3.1. If X is the one-point space, spectral sequence IV is trivial, but spectral sequence V is not trivial in general and does not reduce to the preceding ones,[12]

[12] the initial term of this spectral sequence denoted as in the preceding footnote is $\mathrm{Ext}^p_{\mathbf{O}}(A,H^q(G,B))$.

5.6.4 EXAMPLE. Assume that G is a group of automorphisms of a holomorphic variety, or more generally of a "holomorphic space" X, and that $\mathbf{O}$ is the sheaf of germs of holomorphic functions on X. If A and B are two coherent analytic sheaves on which G operates, since every analytic sheaf that is an extension of A by B is coherent, it follows that the classes of coherent $G-\mathbf{O}$-sheaves that are extensions of A by B correspond to the elements of the complex vector space $\mathrm{Ext}^1_{\mathbf{O},G}(X;A,B)$. Moreover, if n is a positive integer, there is a natural correspondence between the holomorphic fibered vector spaces E whose fibers are $\mathbf{C}^n$ and the coherent algebraic sheaves M over X which are locally isomorphic to $\mathbf{O}^n$; to E corresponds the sheaf $\mathbf{O}(E)$ of germs of holomorphic sections of E, and to M corresponds the holomorphic fibre space whose fiber over $x \in X$ is $M(x) \otimes_{\mathbf{O}(x)} \mathbf{C}$ ($\mathbf{C}$ being considered a module over the augmented algebra $\mathbf{O}(x)$). We can immediately see that in this correspondence, the holomorphic fibered vector spaces with fibers $\mathbf{C}^n$ which admit G as a group of operators correspond to those $G-\mathbf{O}$-modules which are, as $\mathbf{O}$-modules, locally isomorphic to $\mathbf{O}^n$. It follows that the classes of extensions of a holomorphic fiber space E with operators by another one F, naturally correspond to the elements of $\mathrm{Ext}^1_{\mathbf{O},G}(X;\mathbf{O}(E),\mathbf{O}(F))$, which by virtue of Corollary 1 of Theorem 5.6.3, is isomorphic to the vector space $H^1(X;G,\mathbf{O}(E'\otimes F))$ (E' denoting the fiber space dual to E). We can go even further if G is a *discontinuous* group: since we are in characteristic 0, it follows from Corollary 1 of Theorem 5.3.1 that $H^1(X;G,L) \cong H^1(Y,L^G)$ for every G-sheaf L of vector spaces. Moreover, if L is a coherent analytic sheaf, the same is true of L^G (see [5, Chapter 12)]), specifically $\mathbf{O}(E'\otimes F)^G$ is a coherent analytic sheaf over Y. We conclude, for example, from [7] that if Y is compact, then the vector space of the classes of holomorphic fibered vector G-spaces that are extensions of E by F has finite dimension. (In fact, spectral sequence I shows that if Y is compact and A and B are coherent analytic sheaves with operators, then the $\mathrm{Ext}^q_{\mathbf{O},G}(X;A,B)$ have finite dimension.) There are analogous results in abstract algebraical geometry, except that in that case, we can write $H^*(X;G,L) \cong H^*(Y.L^G)$ only if, for example, the order of G is prime to the characteristic, or if G operates "without fixed points".

5.7 Introduction of Φ families

Let Φ be a cofilter of closed subsets of the space X. Consider the functor

$$\Gamma^G_\Phi(A) = \Gamma_\Phi(A)^G \tag{5.7.1}$$

on the category $\mathbf{C}^{X(G)}$ of abelian G-sheaves on X. We set

$$H^p_\Phi(X;G,A) = R^p\Gamma^G_\Phi(A) \tag{5.7.2}$$

If Φ is the cofilter consisting of all the closed subsets of X, we get the functors $H^n(X;G,A)$ from 5.2. Let Φ' be the set of closed subsets of X contained in a G-invariant closed subset

$F \in \Phi$. Then it is clear that Φ' is a cofilter of closed subsets of X, and that $\Gamma_\Phi^G = \Gamma_{\Phi'}^G$. We will therefore henceforth assume that $\Phi = \Phi'$, or, what comes to the same thing, that Φ *is the set of closed subsets $F \subseteq X$ such that $\overline{f(F)} \in \Psi$, where Ψ is a cofilter of closed subsets of Y.* Then we have the natural equivalences

$$\Gamma_\Phi^G \cong \Gamma_\Psi f_*^G \cong \Gamma^G \Gamma_\Phi \tag{5.7.3}$$

Since f_*^G sends injectives of $\mathbf{C}^{X(G)}$ into flabby sheaves, and thus Γ_Ψ-acyclic sheaves, the first equivalence also gives *a cohomological spectral functor on $\mathbf{C}^{X(G)}$ abutting on the graded functor $(H^n(X; G, A))$ whose initial term is:*

$$\mathrm{I}_2^{p,q}(A) = H_\Psi^p(Y, \mathbf{H}^q(G, A)) \tag{5.7.4}$$

The second spectral sequence of Theorem 5.2.1 can no longer be generalized as it stands because we cannot say in general that if B is injective in $\mathbf{C}^{X(G)}$, then $\Gamma_\Phi(B)$ is a Γ^G-acyclic G-module. *Now assume that every set in Ψ has a neighborhood that belongs to Ψ.* For every open set $U \subseteq Y$, let $A_U = A_{f^{-1}(U)}$ be the abelian G-sheaf on X which vanishes on $\complement f^{-1}(U)$ and coincides with A on $f^{-1}(U)$. We have $\Gamma_\Phi(A) = \varinjlim \Gamma_X(A_U)$, where the inductive limit is taken over the filtered set of open subsets $U \subseteq Y$, such that $\overline{(U)} \in \Psi$. Since the functions $\Gamma_X(A_U) \longrightarrow \Gamma_X(A_V)$, for $U \subseteq V$, are injective, the functor Γ^G passes to the inductive limit and we get a natural equivalence:

$$\Gamma_\Phi^G(A) \cong \varinjlim \Gamma_X^G(A_U) \tag{5.7.5}$$

whence, as in Corollary 1 of Proposition 3.10.1:

$$H_\Phi^n(X; G, A) \cong \varinjlim H^n(X; G, A_U) \tag{5.7.6}$$

Thus for every U, $H^*(X; G, A_U)$ is the abutment of a cohomological spectral sequence $\mathrm{II}(A, U)$ whose initial term is

$$\mathrm{II}(A, U)_2^{p,q} = H^p(G, H^q(X, A_U))$$

and as U varies, the spectral sequence forms on inductive system. We then conclude from (5.7.6) that $H_\Phi^*(X; G, A)$ is *the abutment of a cohomological spectral sequence whose initial term is*

$$\mathrm{II}_2^{p,q}(A) = \varinjlim_U H^p(G, H^q(X, A_U)) \tag{5.7.7}$$

Of course, this spectral sequence is also a spectral functor in A. In the most important cases (which we will specify below), we will be able to exchange the symbols $H^p(G, -)$ and

$\varinjlim$, and thus, since $\varinjlim H^q(X, A_U) = H^q(X, A)$ (Corollary 1 of Proposition 3.10.1), we obtain the usual form of the initial term:

$$\mathrm{II}_2^{p,q} = H^p(G, H^q_\Phi(X, A)) \tag{5.7.7$'$}$$

subject to the conditions:

(a) G is a finite group;

(b) G operates trivially on the $H^q(X, A_U)$ and the $H_p(G, Z)$ are of finite type;

(c) the inductive system $(H^q(X, A_U))$ is "essentially constant" for every q, i.e. we can find a cofinal system (U_i) such that the homomorphisms $H^q(X, A_{U_i}) \longrightarrow H^q(X, A_{U_j})$ are isomorphisms.

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